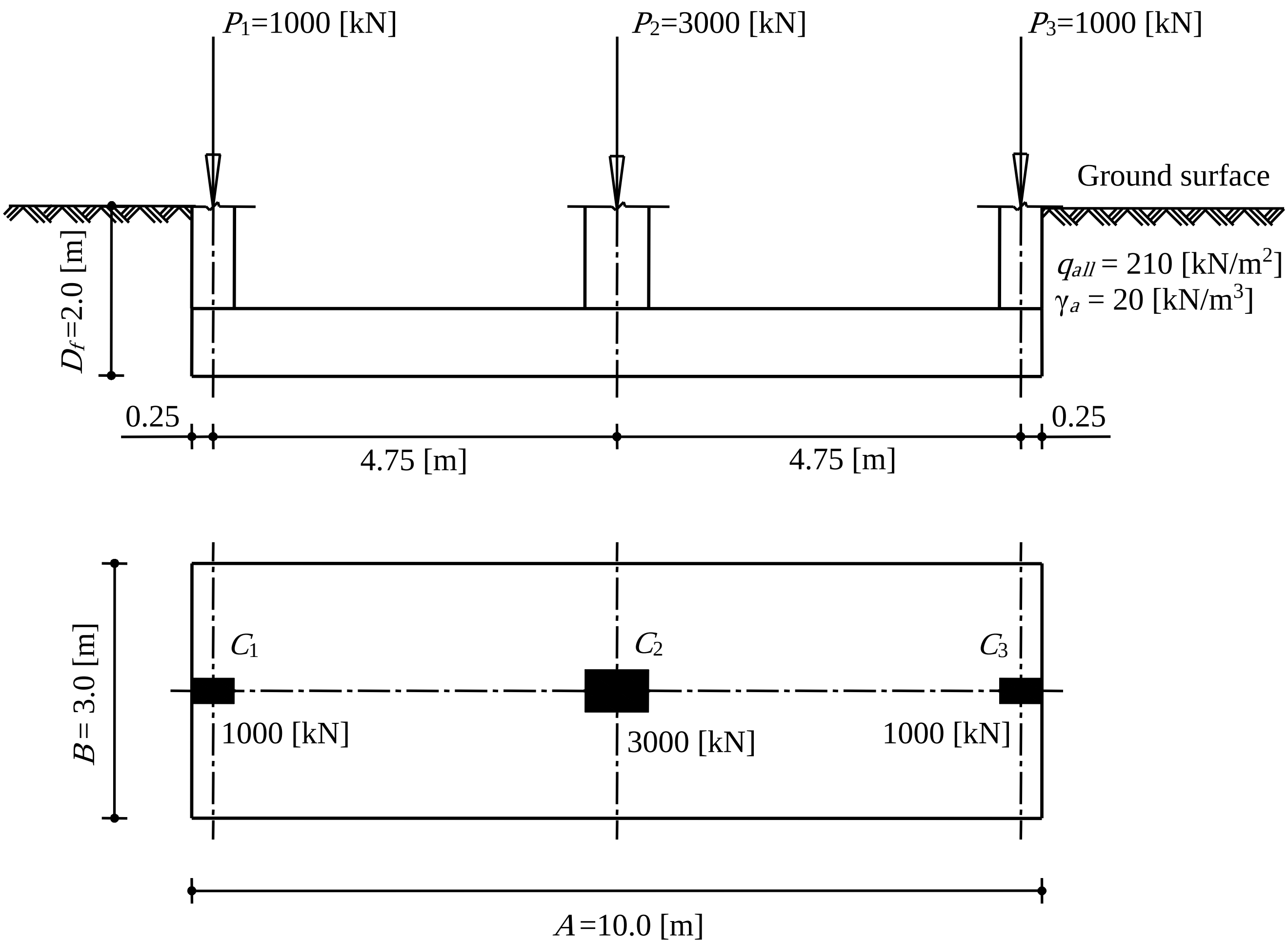
Beam Foundations after *Kany* and *El Gendy* by *GEO Tools* (Analysis and Design)

Part I: Numerical Models



Mahmoud El Gendy
Mohamed El Gendy

Tele.:+1(587) 332-3323
geotec@geotecsoftware.com
www.geotecsoftware.com

2022

Content

Preface

Various problems in Geotechnical Engineering can be investigated by *GEO Tools*. *M. Kany* and (*M. @ A.*) *El Gendy* developed the original version of *GEO Tools* in *ELPLA* package for analyzing elastic foundation. After the death of *Kany* and (*M. & A.*) *El Gendy* further developed the program to meet the needs of the practice.

This book describes the essential methods used in *GEO Tools* for analyzing beam on elastic foundations. *GEO Tools* is a simple user interface program and needs little information to define a problem.

There are three soil models with five methods available in *GEO Tools* for analyzing beam foundations.

10 Analysis of Beam Foundations after *Kany* and *El Gendy*

10.1 Introduction

Different calculation methods are known in the literature for the calculation of shallow foundations. The early one is that assumes a uniform contact pressure distribution under shallow foundations. This assumption is too far from the reality. *Winkler* (1867) and *Zimmermann* (1930) developed the modulus of subgrade method. In the method, the subsoil is simulated by isolated springs. The settlement of the spring is only dependent on the loading at the same point on the subsoil surface at the spring location. This also applies to possible refinements with springs of different stiffness.

However, *Boussinesq* (1885) had already recognized that when the subsoil is loaded at one point, the subsoil also settles outside the load point. Therefore, it does not behave like a spring. Because of this finding, *Ohde* (1942) developed a calculation method for the first time, with which shallow foundations can be analyzed, taking into account the soil structure interaction. This method, which is called modulus of compressibility method, was later further developed by different authors (*Graßhoff* (1966-1978), *Kany* (1974), *Graßhoff*/*Kany* (1992)). *GEO Tools* is based on the modulus of compressibility method after *Kany* (1974) and the modulus of subgrade reaction method after *Kany*/ *El Gendy* (1995). However, some refinements are included, some of which are new and have not yet been dealt with in detail in the literature. It is therefore necessary to explain the calculation methods in more detail than usual in order to be able to check the results and compare them with other results.

10.2 Calculation methods

10.2.1 General

Beam foundations may be analyzed using classical subsoil models. Such as *Winkler*'s model according to *Winkler* (1867), *Graßhoff* (1978) and *Wölfer* (1978) and Continuum model according to *Ohde* (1942), *Graßhoff* (1978) and *Kany* (1974). In addition, cases of small and irregular beam foundations can be analyzed by fewer extensive methods using tables and charts.

It is possible by *GEO Tools* to use the same data for analyzing beam foundations by five different conventional and refined calculation methods based on the three standard subsoil models. The subsoil models for analyzing beam foundations (standard models) available in *GEO Tools* are:

A Simple assumption model
B *Winkler*'s model
C Continuum model

Simple assumption model does not consider the interaction between the beam foundation and the soil. The model assumes a linear distribution of contact pressures beneath the foundation. *Winkler*'s model is the oldest and simplest one that considers the interaction between the beam foundation and the soil. The model represents the soil as elastic springs. Continuum model is the complicated one. The model considers also the interaction between the beam foundation and soil. It represents the soil as a layered continuum medium.

The three standard soil models are described through five different numerical calculation methods. The methods graduate from the simplest one to more complicated one covering the analysis of most common beam foundation problems that may be found in the practice.

According to the three standard soil models (simple assumption model - *Winkler*'s model - Continuum model), five numerical calculation methods are considered to analyze the beam foundation as follows:

1 Linear Contact Pressure
(Simple assumption model)

2 Elastic Beam Foundation using Modulus of Subgrade Reaction by *Kany/ El Gendy* (1995)
(*Winkler*'s model)

3 Elastic Beam Foundation using Modulus of Compressibility by *Kany* (1974)
(Continuum model)

4 Rigid Beam Foundation using Modulus of Compressibility by *Kany* (1972)
(Continuum model)

5 Flexible Beam Foundation using Modulus of Compressibility
(Continuum model)

It is also possible to consider irregular soil layers and the thickness of the base beam that varies in each element. Furthermore, the influence of temperature changes and additional settlement on the beam foundation can be taken into account.

10.2.2 Definition

In the analysis, the beam foundation is divided into equal elements according to Figure 10.1. Using the available five calculation methods, the settlement and the contact pressure can be determined in each element.

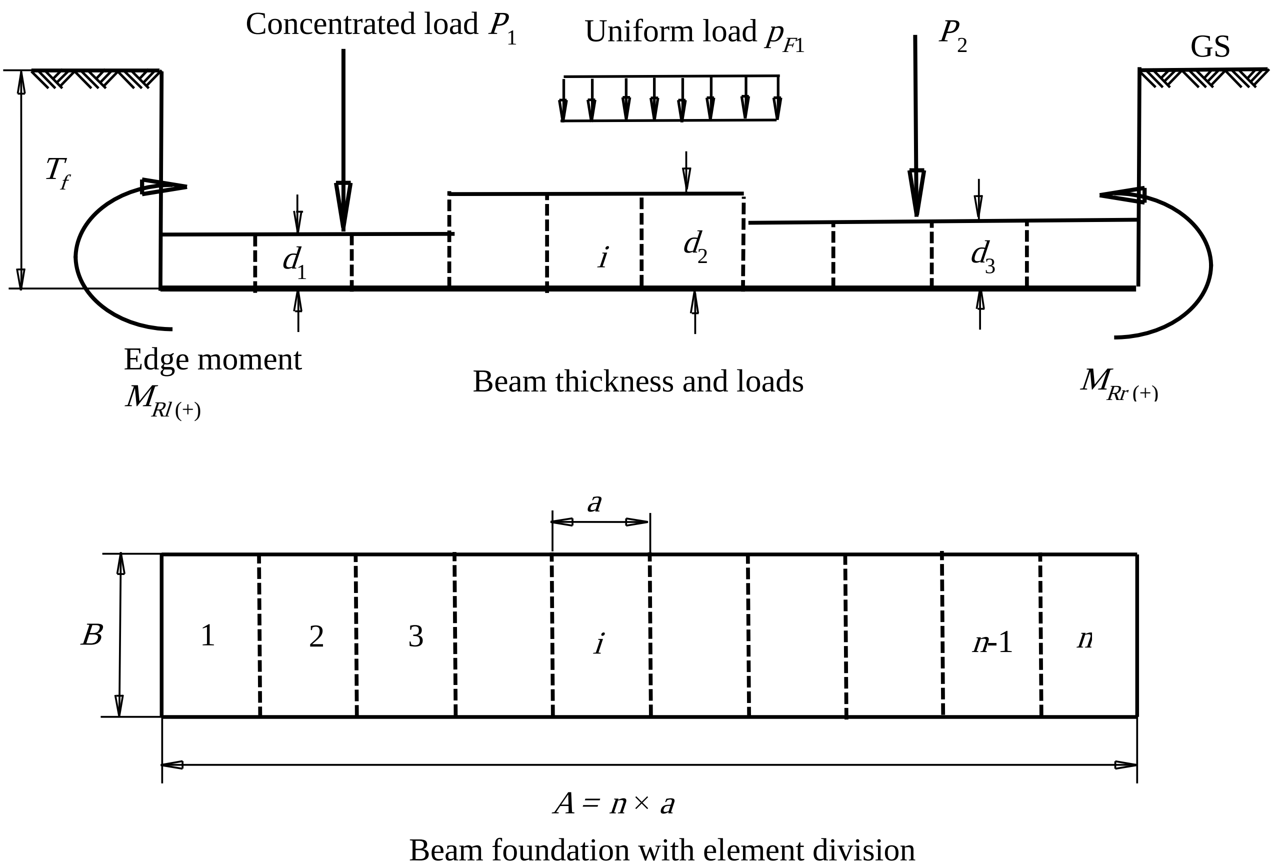


Figure 10.1 Loads, beam thickness und beam foundation with element division

10.3 Linear contact pressure method

This method is the simplest one for determining of the contact pressure distribution under the beam foundation. In the method, it is assumed that the contact pressures are distributed linearly on the bottom of the beam foundations (statically determined) as shown in Figure 10.2. In which the resultant of soil reactions coincides with the resultant of applied loads. Based on *Navier*'s solution, the contact pressure q_i at any point *i* from the geometry centroid of the beam foundation with *N* and M_y is given by

$$q_i = \frac{N}{A_f} + \frac{M_y}{I_y} x_i \tag{10.1}$$

while for a beam foundation without moment $M_y = 0$ or without eccentricity about y-axis, the contact pressure q_i will be uniform under the beam foundation and is given by

$$q_i = \frac{N}{A_f} \tag{10.2}$$

where

N	Sum of all vertical applied loads on the foundation [kN]
x_i	Coordinate of node *i* from the centroidal axis *x* [m]
q_i	Contact pressure at node *i* [kN/m^2]
A_f	Foundation area [m^2]
$M_y=N.e_x$	Moment due to *N* about the *y*-axis [kN.m]
I_y	Moment of inertia of the foundation about the *y*-axis [m^4]

After determining the contact pressure under the beam foundation, the internal forces at the different points of the beam can be calculated.

The assumption of this method is that there is no compatibility between the beam foundation deflection and the soil settlement.

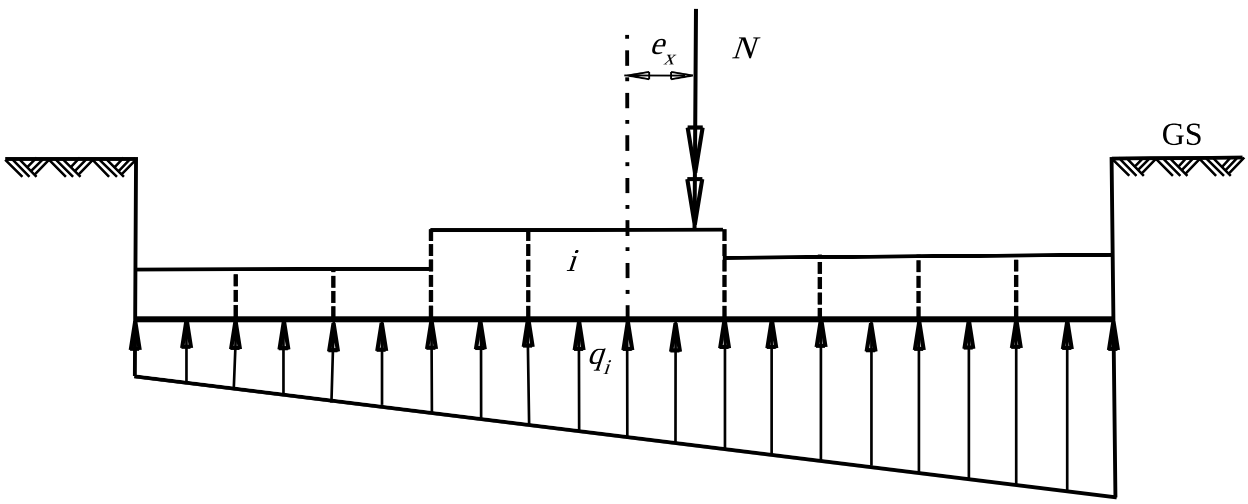


Figure 10.2 Linear contact pressure distribution

10.4 Elastic Beam Foundation by *Kany/ El Gendy* (1995)

The oldest method for the analysis of beam on elastic foundation is the modulus of subgrade reaction, which was proposed by *Winkler* (1867). The assumption of this method is that the soil model is represented by an infinite number of isolated elastic springs. The deflection s_i of the soil medium at any point *i* on the surface is directly proportional to the soil pressure q_i at that point and independent of soil pressures at other locations (Figure 10.3 and Eq. 10.3).

$$q_i = k_i \; s_i \tag{10.3}$$

where

s_i Settlement in element *i* [m]

q_i Contact pressure at element *i* [kN/m^2]

k_i modulus of subgrade reaction at element *i* [kN/m^3]

It should be noticed that k_i is the modulus of subgrade reaction at element ***i***. It may be constant for the whole foundation area or variable from an element to another.

Consider the beam foundation in Figure 10.3. It is necessary to analyze the beam foundation using the method:

Elastic Beam Foundation using Modulus of Subgrade Reaction by *Kany/ El Gendy* (1995).

Assume that the contact pressure distribution is represented by a series of uniform blocks of contact pressures q_n. These values of q_n are the unknowns of the problem.

10.4.1 Settlement s_i

The surface settlement s_i at the center of the element *i* can be written as:

$$\left.\begin{aligned} s_1 &= \frac{q_1}{k_1} \\ s_2 &= \frac{q_2}{k_2} \\ s_3 &= \frac{q_3}{k_3} \\ &\;\vdots \\ s_n &= \frac{q_n}{k_n} \end{aligned}\right\} \tag{10.4}$$

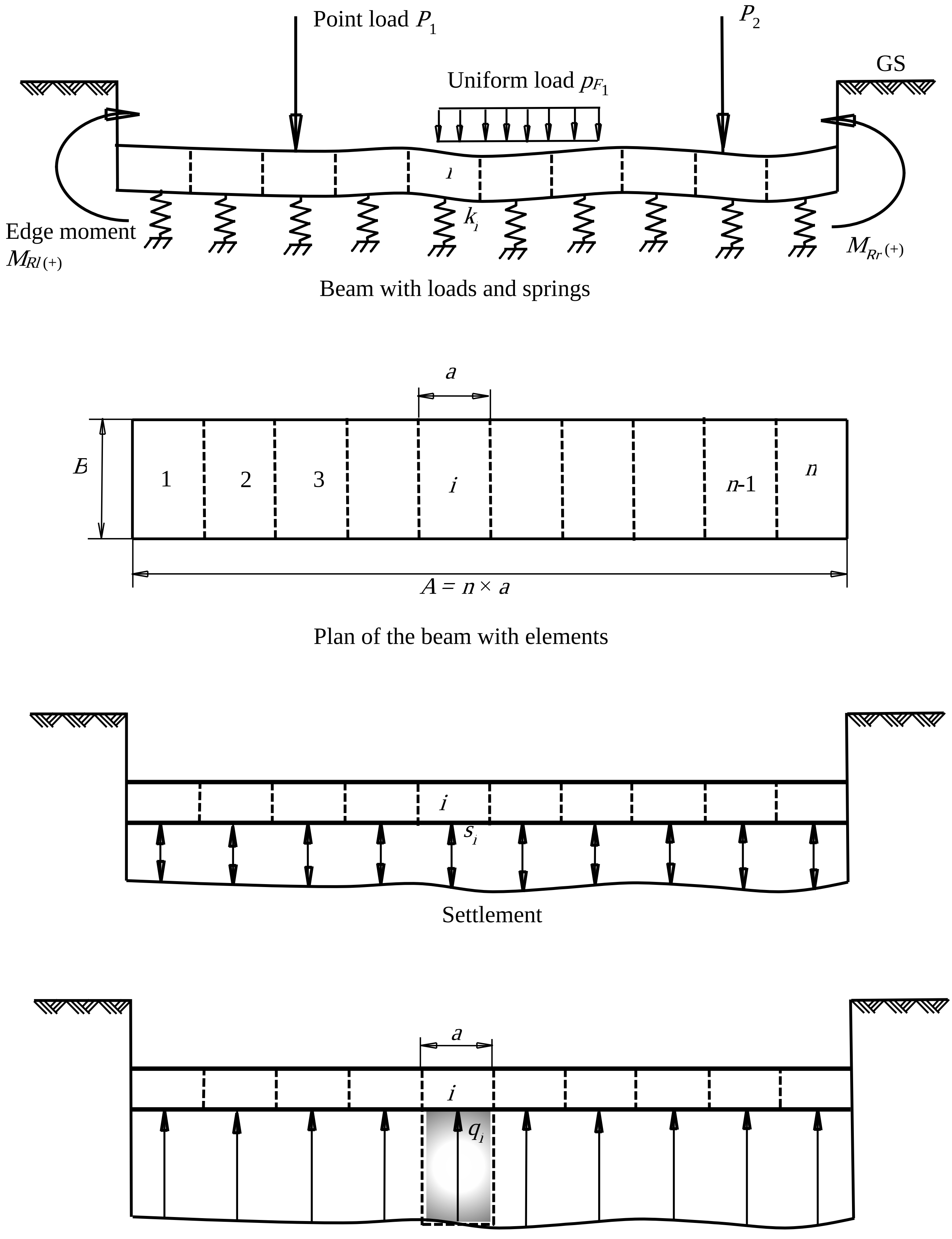


Figure 10.3 Elastic Beam Foundation by *Kany/ El Gendy* (1995)

Equation (10.4) of the settlement at element *i* in general:

$$s_i = \frac{q_i}{k_i} \tag{10.5}$$

10.4.2 Moments M_i

Using *Clapeyron's* three-moment equation, the deflection (settlement) s_i can be related to the moment M_i. For continuity of the elastic curve at the center of element *i*, it is required for elements 2 to *n*-1

$$\left.\begin{aligned}
&- s_1 + 2\, s_2 - s_3 = \left(u_2\, M_{Rl} + v_2\, M_2 + w_2\, M_3\right)\frac{a^2}{6\, E\, I_2}\\
&- s_2 + 2\, s_3 - s_4 = \left(u_3\, M_2 + v_i\, M_3 + w_i\, M_4\right)\frac{a^2}{6\, E\, I_3}\\
&- s_3 + 2\, s_4 - s_5 = \left(u_4\, M_3 + v_4\, M_4 + w_4\, M_5\right)\frac{a^2}{6\, E\, I_4}\\
&.\\
&.\\
&.\\
&- s_{n-2} + 2\, s_{n-1} - s_n = \left(u_{n-1} M_{n-2} + v_{n-1}\, M_{n-1} + w_{n-1}\, M_n\right)\frac{a^2}{6\, E\, I_{n-1}}
\end{aligned}\right\} \tag{10.6}$$

In general:

$$- s_{i-1} + 2\, s_i - s_{i+1} = \left(u_i\, M_{i-1} + v_i\, M_i + w_i\, M_{i+1}\right)\frac{a^2}{6\, E\, I_i} \tag{10.7}$$

where u_i, v_i and w_i are stiffness influence coefficients and are given by

$$u_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i-1}}\right),$$

$$v_i = \frac{1}{4}\left(\frac{I_i}{I_{i-1}} + 14 + \frac{I_i}{I_{i+1}}\right),$$

$$w_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i+1}}\right)$$

and I_i [m^4] is the moment of inertia for cross section element *i* and is given by

$$I_i = \frac{B\, d_i^3}{12}$$

The moment M_i of external forces at the center of element i can be written as:

$$\left.\begin{aligned}
M_2 &= M_{Rl} + a^2 B q_1 - M_2^{(l)} \\
M_3 &= M_{Rl} + 2a^2 B q_1 + a^2 B q_2 - M_3^{(l)} \\
M_4 &= M_{Rl} + 3a^2 B q_1 + 2a^2 B q_2 + a^2 B q_3 - M_4^{(l)} \\
&. \\
&. \\
&. \\
M_{n-1} &= M_{Rl} + (n-1) q_1 a^2 B + (n-2) q_2 a^2 B + (n-3) q_2 a^2 B + \ldots - M_{n-1}^{(l)}
\end{aligned}\right\} \qquad (10.8)$$

or in general:

$$M_i = M_{Rl} + a^2 B \sum_{j=1}^{i} (i - j)\, q_j - M_i^{(l)} \qquad (10.9)$$

where $M_i^{(l)}$ is the external moment due to external loads acting on the center of element i.

10.4.3 Contact pressures q_i for general case

By eliminating s_i and M_i from Eqns. 10.5, 10.7 and 10.9, the following equation can be obtained:

$$\left(\frac{1}{k_{i+1}}\right) q_{i+1} - \left(\frac{2}{k_i} - \frac{\alpha_i}{6} w_i\right) q_i + \left(\frac{1}{k_{i-1}} + \frac{\alpha_i}{6}\left(v_i + 2 w_i\right)\right) q_{i-1} + \frac{\alpha_i}{6}\left(\sum_{j=1}^{i-2}\left[\left(i - j - 1\right) u_i + \left(i - j\right) v_i + \left(i - j + 1\right) w_i\right] q_j\right) = R_i$$

(10.10)

where

$$\alpha_i = \frac{a^4 B}{E I_i}$$

$$R_i = \left(u_i M^{(l)}_{i-1} + v_i M^{(l)}_{i} + w_i M^{(l)}_{i+1}\right) \frac{a^2}{6 E I_i} \qquad (10.11)$$

Equation 10.10 can be applied at elements 2 to n-2, therefore two further equations are required to obtain the n unknown contact pressures q_1 to q_n. This can be done by considering the overall equilibrium of the vertical forces and moments of the beam foundation.

10.4.3.1 Equilibrium of the vertical forces:

The resultant N due to external vertical forces acting on the beam must be equal to the sum of contact forces

$$\left.\begin{array}{l} \Sigma V = 0 \\ aB(q_1 + q_2 + q_3 + ... + q_n) - \Sigma P = 0 \end{array}\right\} \qquad (10.12)$$

10.4.3.2 Equilibrium of the moments about y-axis:

Furthermore, the moments around the *y*-axis must be in equilibrium

$$\left.\begin{array}{l} \Sigma M = 0 \\ \frac{(2n-1)}{2} q_1 a^2 B + \frac{(2n-3)}{2} q_2 a^2 B + \frac{(2n-5)}{2} q_3 a^2 B + ... + \frac{1}{2} q_n a^2 B - M_{Rl} + M_{Rr} - \Sigma M^{(l)} = 0 \end{array}\right\} \qquad (10.13)$$

Equation 10.10 to Eq. 10.13 can then be used to obtain the unknown soil contact pressures q_n for any arbitrary external loading condition.

Once the contact pressures q_i are obtained at the various sections, then the internal forces in the beam can be calculated.

10.4.4 Contact pressures q_i for constants k_i and I_i

For constant $k_i = k$, $I_i = I$, $u_i = 1$, $v_i = 4$, $w_i = 1$ and $\alpha_i = \alpha$. Then Eq. 10.10 becomes

$$\left(\frac{1}{k}\right) q_{i+1} - \left(\frac{2}{k} - \frac{\alpha}{6}\right) q_i + \left(\frac{1}{k} + \alpha\right) q_{i-1} + \alpha \left(\sum_{j=1}^{i-2} (i-j) q_j\right) = R_i \qquad (10.14)$$

where

$$R_i = \left(M^{(l)}_{i-1} + 4 M^{(l)}_i + M^{(l)}_{i+1}\right) \frac{a^2}{6E I_i}$$

For i= 2 to i= n-2, Eq. 10.14 becomes:

$$
\left.
\begin{aligned}
&\left(\frac{1}{k}+\alpha\right)q_1 - \left(\frac{2}{k}-\frac{\alpha}{6}\right)q_2 + \left(\frac{1}{k}\right)q_3 = R_2 \\
&2\alpha\, q_1 + \left(\frac{1}{k}+\alpha\right)q_2 - \left(\frac{2}{k}-\frac{\alpha}{6}\right)q_3 + \left(\frac{1}{k}\right)q_4 = R_3 \\
&3\alpha\, q_2 + 2\alpha\, q_2 + \left(\frac{1}{k}+\alpha\right)q_3 - \left(\frac{2}{k}-\frac{\alpha}{6}\right)q_4 + \left(\frac{1}{k}\right)q_5 = R_4 \\
&. \\
&. \\
&. \\
&(n-3)\alpha q_1 + (n-4)\alpha q_2 + \ldots + \left(\frac{1}{k}+\alpha\right)q_{n-2} - \left(\frac{2}{k}-\frac{\alpha}{6}\right)q_{n-1} + \left(\frac{1}{k}\right)q_n = R_{n-1}
\end{aligned}
\right\} \qquad (10.15)
$$

10.4.5 Contact pressures q_i for a symmetrical case

For a symmetrical beam foundation with n=8 elements, the number of equations can be reduced to 4. Due to the symmetry $q_1=q_8$, $q_2= q_7$, $q_3= q_6$ and $q_4=q_5$.

Equation 10.12 and Eq. 10.15 become:

$$
\left.
\begin{aligned}
&q_1 + q_2 + q_3 + q_4 = \frac{\Sigma P}{aB} \\
&(1+\alpha k)q_1 - \left(2-\frac{\alpha k}{6}\right)q_2 + q_3 = kR_2 \\
&2\alpha k\, q_1 + (1+\alpha k)q_2 - \left(2-\frac{\alpha k}{6}\right)q_3 + q_4 = kR_3 \\
&3\alpha k\, q_1 + 2\alpha k\, q_2 + (1+\alpha k)q_3 - \left(1-\frac{\alpha k}{6}\right)q_4 = kR_4 .
\end{aligned}
\right\} \qquad (10.16)
$$

In matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (1+\alpha k) & -\left(2-\frac{\alpha k}{6}\right) & 1 & 0 \\ 2\alpha k & (1+\alpha k) & -\left(2-\frac{\alpha k}{6}\right) & 1 \\ 3\alpha k & 2\alpha k & (1+\alpha k) & -\left(1-\frac{\alpha k}{6}\right) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \frac{\Sigma P}{aB} \\ kR_2 \\ kR_3 \\ kR_4 \end{Bmatrix} \tag{10.17}$$

10.4.6 Determining modulus of subgrade reaction

The modulus of subgrade reaction k_i can be defined by the user or determined by the settlement calculation. These two options for defining the modulus of subgrade reaction are described in the next sections.

10.4.6.1 Modulus k is calculated from soil layers

In this case, variable modulus of subgrade reaction k_i are calculated at different elements i.

i) First, linear distribution of contact pressure $q^{(o)}$ on the bottom of the beam foundation is assumed as (Figure 10.4)

$$q^{(o)} = \frac{N}{A_f} + \frac{M_y}{I_y} x \tag{10.18}$$

ii) For a set of n elements, the soil settlement s_i at element i due to contact pressure is obtained from the following formula according to *Ohde* (1942)

$$s_i = \sum_{j=1}^{n} c_{i,j}\, q_j \tag{10.19}$$

where $c_{i,j}$ is the flexibility coefficients of point i due to a unit loading on element j

iii) From the calculated soil settlement s_i and contact pressure q_i, the modulus of subgrade reaction for all elements k_i is computed according to Figure 10.4, *Winkler* (1867)

$$k_i = \frac{q_i}{s_i} \tag{10.20}$$

iv) The mean modulus of subgrade reaction k_m for the whole beam is then given by

$$k_{sm} = \frac{1}{n}\sum_{i=1}^{n} k_i \tag{10.21}$$

where n is the number of elements

The further calculation is carried out using the Elastic Beam by Modulus of Subgrade Reaction after *Kany/ El Gendy* (1995).

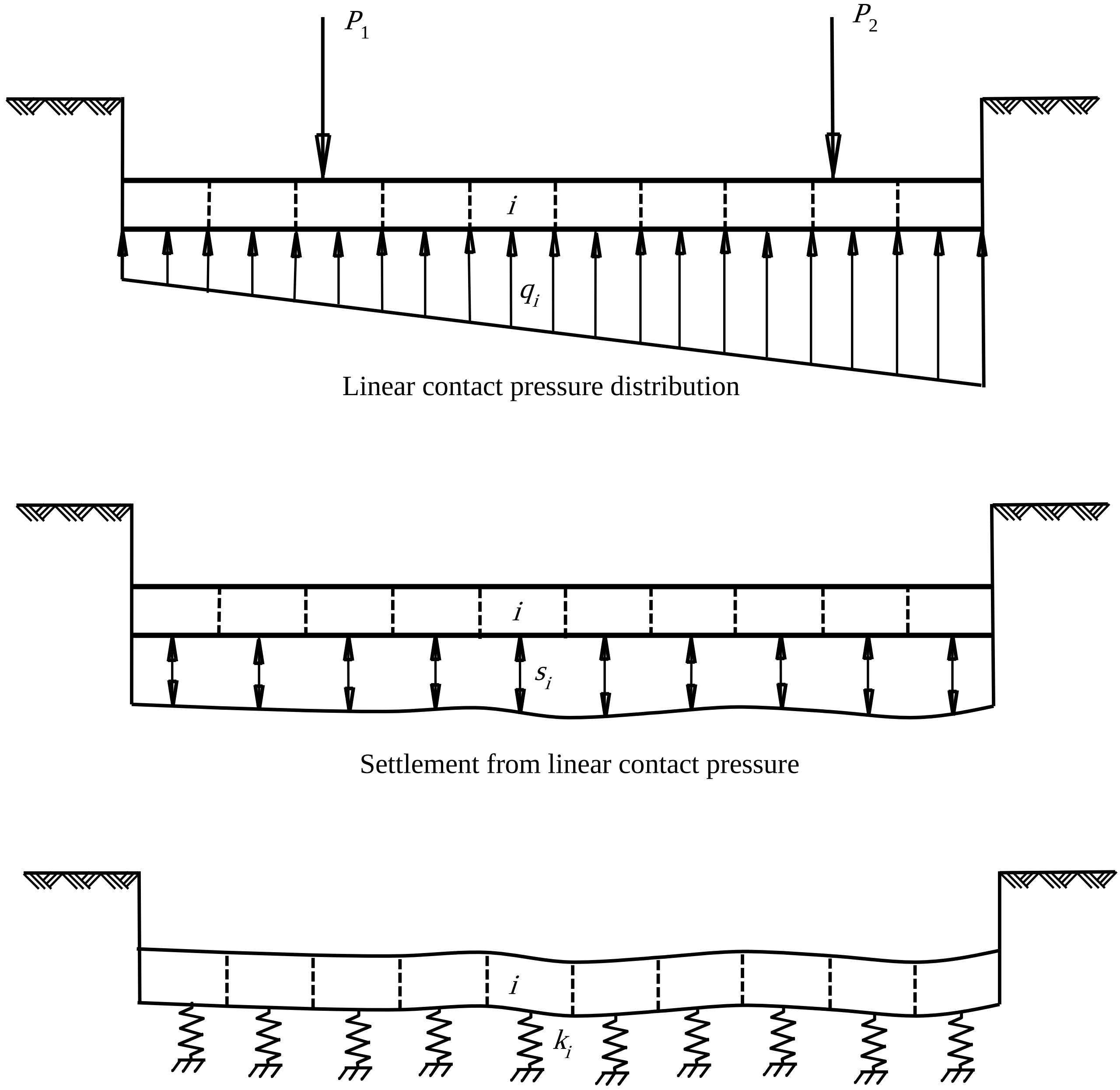


Figure 10.4 Calculation of modulus of subgrade reaction

10.4.6.2 Modulus k is defined by the user

The user can define the modulus of subgrade reaction *k*, which is constant or variable from element to element.

10.5 Elastic Beam Foundation by *Kany* (1974)

Continuum model was first proposed by *Ohde* (1942), which based on the settlement occurs not only under the loaded area but also outside it. Otherwise, the settlement at any point is affected by loads at all the other elements. Using this concept, influence lines of the settlement due to loaded areas on the surface can be constructed as shown in Figure 10.5.

From influence lines of the settlement of Figure 10.5, the settlement s_i at the center of the element i can be obtained from:

$$\left.\begin{array}{lllllllllll} s_1 & = & s_{1,1} & + & s_{1,2} & + & s_{1,3} & + & \ldots & + & s_{1,n} \\ s_2 & = & s_{2,1} & + & s_{2,2} & + & s_{2,3} & + & \ldots & + & s_{2,n} \\ s_3 & = & c_{3,1} & + & s_{3,2} & + & s_{3,3} & + & \ldots & + & s_{3,n} \\ \\ \\ s_n & = & s_{n,1} & + & s_{n,2} & + & s_{n,3} & + & \ldots & + & s_{n,n} \end{array}\right\} \tag{10.22}$$

where $s_{i,j}$ [m] is the settlement of point i due to a uniform load q_j [kN/m^3] on element j

Since the settlement $s_{i,j}$ can be obtained as a function of a uniform load q_j on the surface, the settlement with a flexibility coefficient can be written as follows:

$$s_{i,j} = c_{i,j} q_j \tag{10.23}$$

where $c_{i,j}$ [m^3/kN] is the flexibility coefficient of point i due to a uniform load q_j at element j.

Equation (10.18) can be rewritten with flexibility coefficients as:

$$\left.\begin{array}{lllllllllll} s_1 & = & c_{1,1}\,q_1 & + & c_{1,2}\,q_2 & + & c_{1,3}\,q_3 & + & \ldots & + & c_{1,n}\,q_n \\ s_2 & = & c_{2,1}\,q_1 & + & c_{2,2}\,q_2 & + & c_{2,3}\,q_3 & + & \ldots & + & c_{2,n}\,q_n \\ s_3 & = & c_{3,1}\,q_1 & + & c_{3,2}\,q_2 & + & c_{3,3}\,q_3 & + & \ldots & + & c_{3,n}\,q_n \\ \\ \\ s_n & = & c_{n,1}\,q_1 & + & c_{n,2}\,q_2 & + & c_{n,3}\,q_3 & + & \ldots & + & c_{n,n}\,q_n \end{array}\right\} \tag{10.24}$$

10.5.1 Settlements s_i

Now, consider the beam foundation shown in Figure 10.6 is divided into equal elements, each of a length a. Assume that the contact pressure distribution can be approximated by a series of uniform blocks of contact pressures. These contact pressures q_n are selected as the unknowns of the problem.

For a beam of equal elements, the flexibility coefficients then becomes $c_{i,j} = c_{j,i} = c_i$ and $c_{i,i} = c_0$.

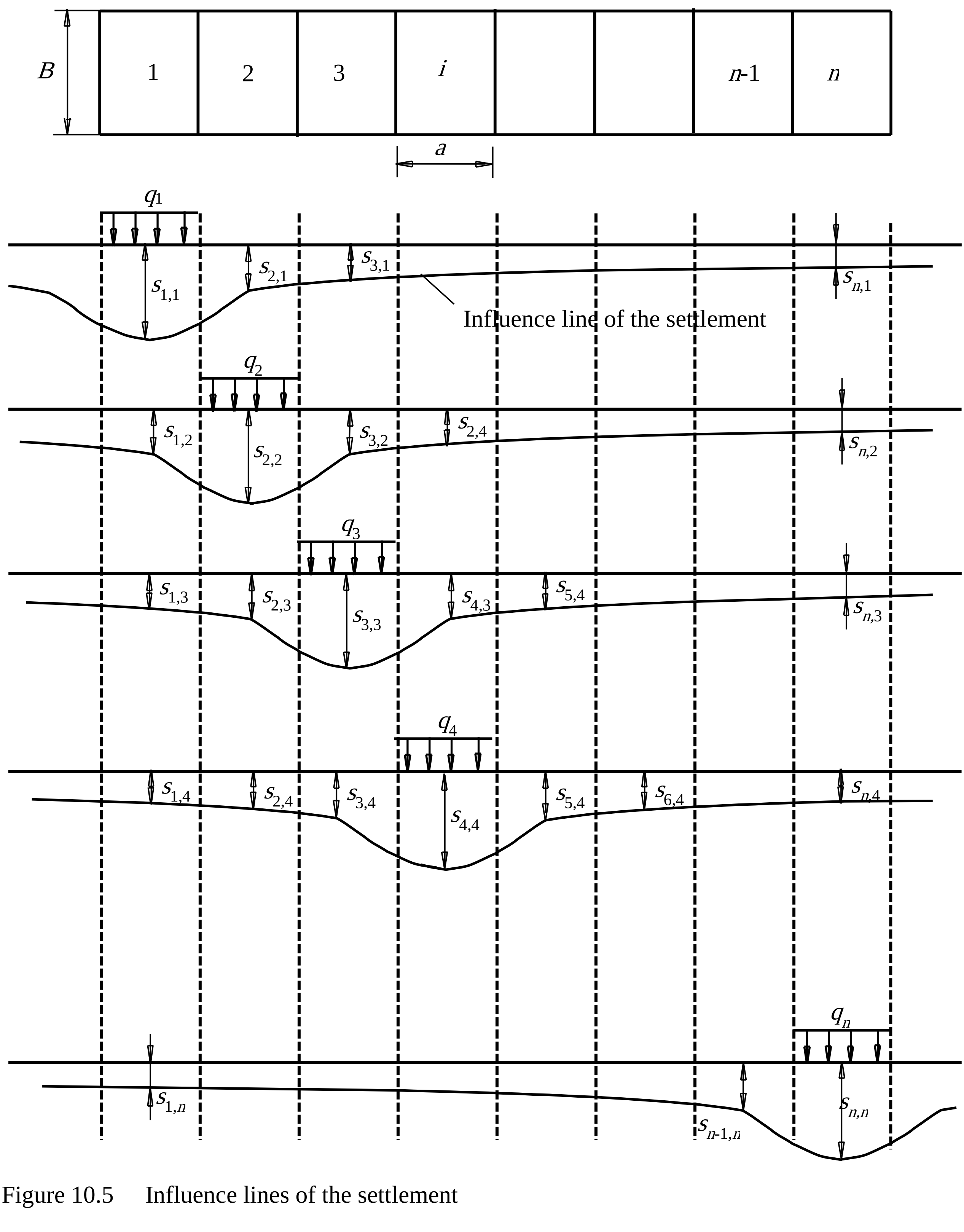


Figure 10.5 Influence lines of the settlement

Therefore, the settlement s_i at the center of the element *i* can be written as:

$$\left.\begin{array}{lllllllllll} s_1 & = & c_0\, q_1 & + & c_1\, q_2 & + & c_2\, q_3 & + & \dots & + & c_{n-1}\, q_n \\ s_2 & = & c_1\, q_1 & + & c_0\, q_2 & + & c_1\, q_3 & + & \dots & + & c_{n-2}\, q_n \\ s_3 & = & c_2\, q_1 & + & c_1\, q_2 & + & c_0\, q_3 & + & \dots & + & c_{n-3}\, q_n \\ \\ \\ s_n & = & c_{n-1}\, q_1 & + & c_{n-2}\, q_2 & + & c_{n-3}\, q_3 & + & \dots & + & c_0\, q_n \end{array}\right\} \qquad (10.25)$$

In general:

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j \qquad (10.26)$$

10.5.2 Moments M_i

Using *Clapeyron's* three-moment equation, the deflection of the beam (= settlement of the soil) s_i can be related to the moment M_i. For continuity of the elastic curve at the center of element *i*, it is required for elements 2 to *n*-1

$$\left.\begin{array}{l} -s_1 + 2\, s_2 - s_3 = \left(u_2\, M_{Rl} + v_2\, M_2 + w_2\, M_3\right)\dfrac{a^2}{6\, E\, I_2} \\ -s_2 + 2\, s_3 - s_4 = \left(u_3\, M_2 + v_i\, M_3 + w_i\, M_4\right)\dfrac{a^2}{6\, E\, I_3} \\ -s_3 + 2\, s_4 - s_5 = \left(u_4\, M_3 + v_4\, M_4 + w_4\, M_5\right)\dfrac{a^2}{6\, E\, I_4} \\ . \\ . \\ . \\ -s_{n-2} + 2\, s_{n-1} - s_n = \left(u_{n-1} M_{n-2} + v_{n-1}\, M_{n-1} + w_{n-1}\, M_n\right)\dfrac{a^2}{6\, E\, I_{n-1}} \end{array}\right\} \qquad (10.27)$$

In general:

$$-s_{i-1} + 2\, s_i - s_{i+1} = \left(u_i\, M_{i-1} + v_i\, M_i + w_i\, M_{i+1}\right)\frac{a^2}{6\, E\, I_i} \qquad (10.28)$$

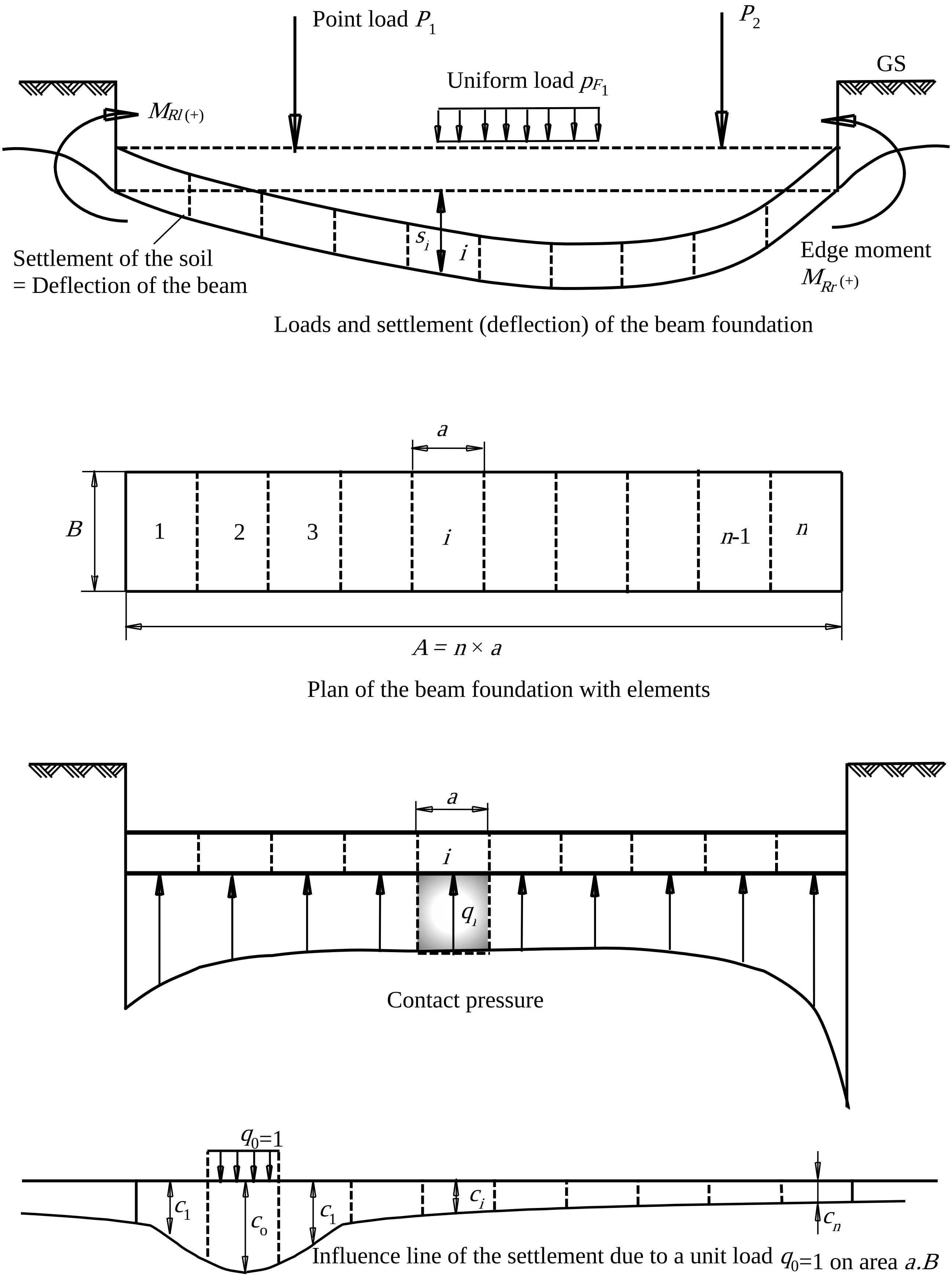


Figure 10.6 Elastic Beam Foundation by *Kany* 1974

where u_i, v_i and w_i are stiffness influence coefficients and are given by

$$u_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i-1}}\right),$$

$$v_i = \frac{1}{4}\left(\frac{I_i}{I_{i-1}} + 14 + \frac{I_i}{I_{i+1}}\right),$$

$$w_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i+1}}\right)$$

and I_i [m^4] is the moment of inertia for cross section element i and is given by

$$I_i = \frac{B\, d_i^3}{12}$$

The moment M_i of external forces at the center of element i can be written as:

$$\left.\begin{aligned}
&M_2 = M_{Rl} + a^2 B q_1 - M_2^{(l)}\\
&M_3 = M_{Rl} + 2a^2 B q_1 + a^2 B q_2 - M_3^{(l)}\\
&M_4 = M_{Rl} + 3a^2 B q_1 + 2a^2 B q_2 + a^2 B q_3 - M_4^{(l)}\\
&.\\
&.\\
&.\\
&M_{n-1} = M_{Rl} + (n-1) q_1 a^2 B + (n-2) q_2 a^2 B + (n-3) q_2 a^2 B + \ldots - M_{n-1}^{(l)}
\end{aligned}\right\} \qquad (10.29)$$

In general:

$$M_i = M_{Rl} + a^2 B \sum_{j=1}^{i} (i - j)\, q_j - M_i^{(l)} \qquad (10.30)$$

where $M_i^{(l)}$ is the external moment due to external loads acting on the center of element i.

10.5.3 Contact pressures q_i for a general case

By eliminating s_i and M_i from Equations (10.26), (10.28) and (10.30), the following equation for i= 2 to i= n-2 can be obtained:

$$\sum_{j=2}^{i}\left(C_{i-j+1} + \left[(i-j)u_i + (i-j+1)v_i + (i-j+2)w_i\right]\frac{\alpha_i}{6}\right)q_j + \left(C_0 + w_i \frac{\alpha_i}{6}\right)q_i + \sum_{j=1}^{n-i} C_j\, q_{i+j} = R_i \qquad (10.31)$$

The constant C_i is related to c_i where

$$C_o = 2(c_1 - c_0), \quad C_1 = c_0 - 2c_1 + c_2, \quad C_2 = c_1 - 2c_2 + c_3, \ldots C_{n-2} = c_{n-3} - 2c_{n-2} + c_{n-1}$$

and

$$\alpha_i = \frac{a^4 B}{E\, I_i}$$

$$R_i = \left(u_i\, M^{(l)}{}_{i\text{-}1} + v_i\, M^{(l)}{}_i + w_i\, M^{(l)}{}_{i+1}\right)\frac{a^2}{6E\, I_i} \qquad (10.32)$$

Equation 10.31 can be applied at elements 2 to n-2, therefore two further equations are required to obtain the n unknown contact pressures q_1 to q_n. This can be done by considering the overall equilibrium of the vertical forces and moments of the beam foundation.

10.5.3.1 Equilibrium of the vertical forces:

The resultant N due to external vertical forces acting on the raft must be equal to the sum of contact forces

$$\left.\begin{array}{l} \Sigma V = 0 \\ aB\left(q_1 + q_2 + q_3 + ... + q_n\right) - \Sigma P = 0 \end{array}\right\} \qquad (10.33)$$

10.5.3.2 Equilibrium of the moments about y-axis:

Furthermore, the moments around the y-axis must be in equilibrium

$$\left.\begin{array}{l} \Sigma M = 0 \\ \frac{(2n-1)}{2} q_1 a^2 B + \frac{(2n-3)}{2} q_2 a^2 B + \frac{(2n-5)}{2} q_3 a^2 B + ... + \frac{1}{2} q_n a^2 B - M_{Rl} + M_{Rr} - \Sigma M^{(l)} = 0 \end{array}\right\} \qquad (10.34)$$

Equation 10.31 to Eq. 10.34 can then be used to obtain the unknown soil contact pressures q_n for any arbitrary external loading condition. Settlements of the soil under the beam foundation can be obtained by substituting the calculated contact pressures in Eq. 10.26.

Once the contact pressures q_i are obtained at the various sections, then the internal forces in the beam can be calculated.

10.5.4 Contact pressures q_i for constant I_i

For constant $I_i = I$, $u_i = 1$, $v_i = 4$, $w_i = 1$ and $\alpha_i = \alpha$. Then Eq. 10.31 becomes

$$\sum_{j=2}^{i}\left(C_{i-j+1} + (i-j+1)\alpha_i\right)q_j + \left(C_0 + \frac{\alpha_i}{6}\right)q_i + \sum_{j=1}^{n-i} C_j q_{i+j} = R_i \tag{10.35}$$

For i= 2 to i= n-2, Eq. 10.35 becomes:

$$\left.\begin{aligned}
&(C_1 + \alpha)q_1 + \left(C_0 + \frac{\alpha}{6}\right)q_2 + C_1 q_3 + C_2 q_4 + \ldots + C_{n-2} q_n = R_2 \\
&(C_2 + 2\alpha)q_1 + (C_1 + \alpha)q_2 + \left(C_0 + \frac{\alpha}{6}\right)q_3 + C_1 q_4 + \ldots + C_{n-3} q_n = R_3 \\
&(C_3 + 3\alpha)q_1 + (C_2 + 2\alpha)q_2 + (C_1 + \alpha)q_3 + \left(C_0 + \frac{\alpha}{6}\right)q_4 + C_1 q_5 \ldots + C_{n-4} q_n = R_4 \\
&. \\
&. \\
&. \\
&(C_{n-2} + (n-2)\alpha)q_1 + (C_{n-3} + (n-3)\alpha)q_2 + \ldots + \left(C_0 + \frac{\alpha}{6}\right)q_{n-1} + \ldots + C_1 q_n = R_{n-1}
\end{aligned}\right\} \tag{10.36}$$

10.5.5 Contact pressures q_i for symmetrical case

For a symmetrical beam foundation with n=8 elements, the number of equations can be reduced to 4. Due to the symmetry $q_1=q_8$, $q_2= q_7$, $q_3= q_6$ and $q_4=q_5$.

Equation 10.33 and Eq. 10.36 becomes:

$$\left.\begin{aligned}
&q_1+q_2+q_3+q_4=\frac{\Sigma P}{aB}\\
&(C_1+C_6+\alpha)q_1+\left(C_0+C_5+\frac{\alpha}{6}\right)q_2+(C_1+C_4)q_3+(C_2+C_3)q_4=R_2\\
&(C_2+C_5+2\alpha)q_1+(C_1+C_4+\alpha)q_2+\left(C_0+C_3+\frac{\alpha}{6}\right)q_3+(C_1+C_2)q_4=R_3\\
&(C_3+C_4+3\alpha)q_1+(C_2+C_3+2\alpha)q_2+(C_1+C_2+\alpha)q_3+\left(C_0+C_1+\frac{\alpha}{6}\right)q_4=R_4
\end{aligned}\right\} \qquad (10.37)$$

In matrix form:

$$\begin{bmatrix}
1 & 1 & 1 & 1\\
(C_1+C_6+\alpha) & \left(C_0+C_5+\frac{\alpha}{6}\right) & (C_1+C_4) & (C_2+C_3)\\
(C_2+C_5+2\alpha) & (C_1+C_4+\alpha) & \left(C_0+C_3+\frac{\alpha}{6}\right) & (C_1+C_2)\\
(C_3+C_4+3\alpha) & (C_2+C_3+2\alpha) & (C_1+C_2+\alpha) & \left(C_0+C_1+\frac{\alpha}{6}\right)
\end{bmatrix}
\begin{Bmatrix} q_1\\ q_2\\ q_3\\ q_4 \end{Bmatrix}
=
\begin{Bmatrix} \frac{\Sigma P}{aB}\\ R_2\\ R_3\\ R_4 \end{Bmatrix} \qquad (10.38)$$

10.6 Rigid beam foundation by *Kany* (1972)

In the case of rigid beam foundations, it is assumed that the beam is so thick that no significant deformations of the beam occur.

In many practical cases, it is convenient to treat the beam as being infinitely rigid, where two conclusions can be drawn concerning beam settlement:

1. If there are no moments $M = 0$ caused by load eccentricity, all points on the beam will go down the same amount s_o.
2. If there are moments $M \neq 0$, the beam will rotate as a rigid body and there will be differential vertical movement between points on the beam, but all points will remain in the same plane.

10.6.1 Case of an eccentric load ($e_x \neq 0$)

For a beam with an eccentric load about *y*-axis (Figure 10.7), the unknowns of the problem are *n* contact pressures q_i, the uniform rigid body translation s_o and the rotation α about *y*-axis.

10.6.1.1 Soil settlements

To formulate the stiffness matrix for analyzing the rigid beam foundation on a layered soil medium (isotropic elastic half-space soil medium may be also applied), consider a set of *n* elements of the beam as shown in Figure 10.7. According to *Kany* (1972), the contact pressure at rigid beam-subsoil interface can be approximated by a series of blocks of uniform stress intensity. The settlement s_i at a soil element *i* due to contact pressures on *n* elements is given by

$$\left.\begin{array}{lllllllllll} s_1 & = & c_0\, q_1 & + & c_1\, q_2 & + & c_2\, q_3 & + & ... & + & c_{n-1}\, q_n \\ s_2 & = & c_1\, q_1 & + & c_0\, q_2 & + & c_1\, q_3 & + & ... & + & c_{n-2}\, q_n \\ s_3 & = & c_2\, q_1 & + & c_1\, q_2 & + & c_0\, q_3 & + & ... & + & c_{n-3}\, q_n \\ & & & & & & & & & & \\ s_n & = & c_{n-1}\, q_1 & + & c_{n-2}\, q_2 & + & c_{n-3}\, q_3 & + & ... & + & c_0\, q_n \end{array}\right\} \tag{10.39}$$

In general:

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j \tag{10.40}$$

where c_{i-j} is the flexibility coefficient of an element *i* due to a unit load q_j at element *j*, [m^3/kN].

Considering the entire beam foundation, Eq. 10.40 is rewritten in matrix form as:

$$\{s\} = [c]\{q\} \tag{10.41}$$

where

$\{s\}$ Vector of settlements

$[c]$ Flexibility matrix

$\{q\}$ Vector of contact pressures

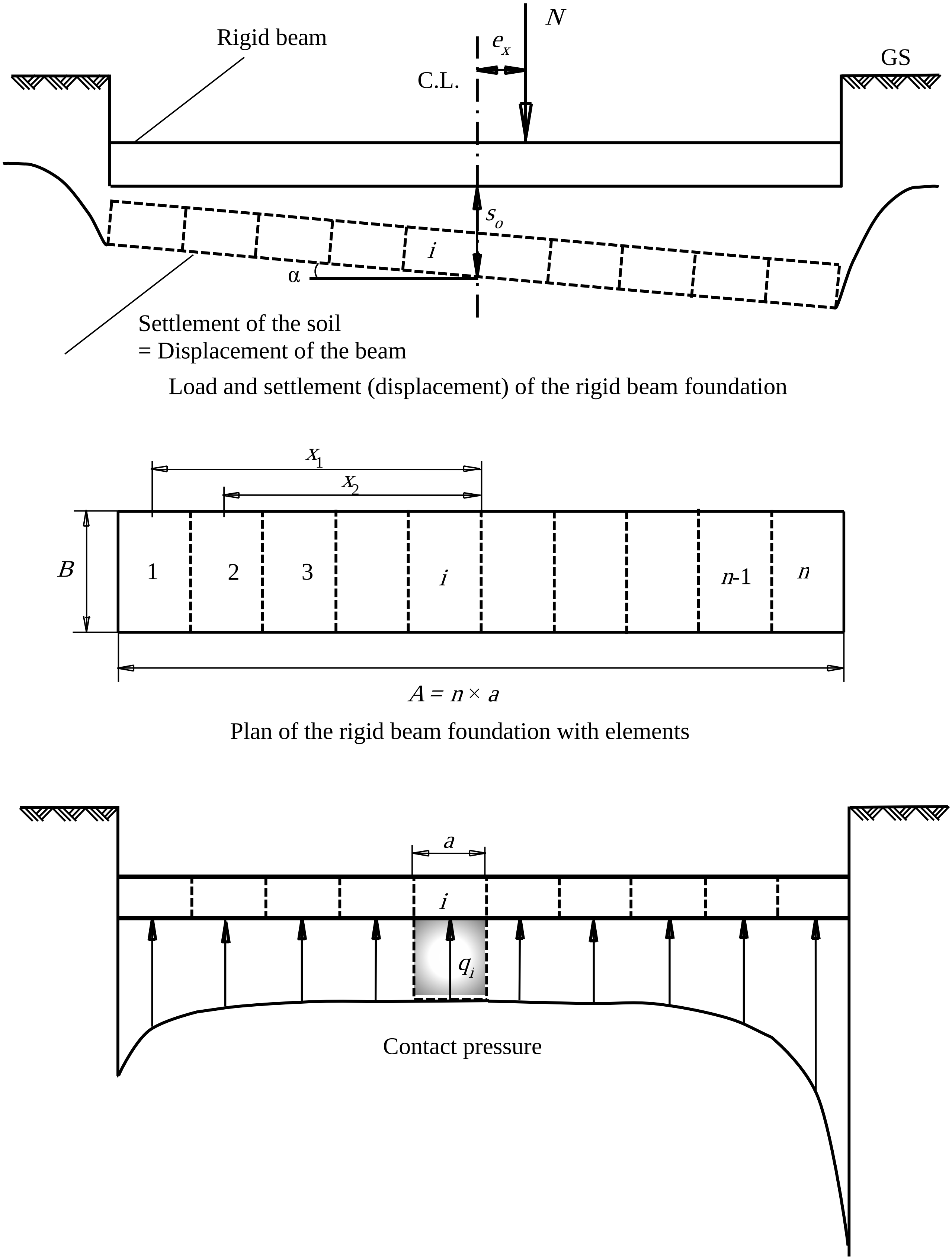


Figure 10.7 Rigid Beam Foundation by *Kany* 1972 (case of an eccentric load)

Inverting the flexibility matrix [*c*], gives the stiffness matrix of the soil [k_s] corresponding to the contact pressures at the *n* elements such that

$$\{q\} = [k_s]\{s\} \tag{10.42}$$

where [*k*]=[*c*]$^{-1}$ is the soil stiffness matrix.

10.6.1.2 Rigid body translation s_o and rotations α

Due to the beam rigidity, the following linear relation expresses the settlement s_i at the center of element *i* that has a distance x_i from the geometry centroid:

$$s_i = s_o + x_i \tan \alpha \tag{10.43}$$

Equation 10.43 is rewritten in matrix form for the entire beam foundation as

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ . \\ . \\ . \\ s_n \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ . & . \\ . & . \\ . & . \\ 1 & x_n \end{bmatrix} \begin{Bmatrix} s_o \\ \tan\alpha \end{Bmatrix} \tag{10.44}$$

Equation 10.44 is simplified to

$$\{s\} = [X]^T \{\Delta\} \tag{10.45}$$

where

{Δ} Vector of the beam deformation from s_o, tan α

$[X]^T$ Geometric matrix of the beam with coordinate *x*

Substituting Eq. 10.45 into Eq. 10.42 gives

$$\{q\} = \{k_s\}[X]^T \{\Delta\} \tag{10.46}$$

Equation 10.46 is a matrix of *n* equations with *n*+2 unknowns, namely the contact pressures q_1 to q_n, the uniform rigid body translation s_o and the rotation α about *y*-axis. Therefore, two further equations are required. This can be done by considering the overall equilibrium of the vertical forces and moments of the beam foundation.

10.6.1.3 Equilibrium of the vertical forces:

The resultant *N* due to external vertical forces acting on the raft must be equal to the sum of contact forces

$$\left.\begin{aligned} &\Sigma V = 0 \\ &N = aB\ (q_1 + q_2 + q_3 + ... + q_n) \end{aligned}\right\} \tag{10.47}$$

10.6.1.4 Equilibrium of the moments about y-axis:

Furthermore, the moment M=$N.e_x$ due to resultant *N* about the *y*-axis must be equal to the sum of moments due to contact forces about that axis

$$\left.\begin{aligned} &\Sigma M = 0 \\ &M = aB(q_1 x_1 + q_2 x_2 + q_3 x_3 + ... + q_n x_n) \end{aligned}\right\} \tag{10.48}$$

Equations 10.47 and 10.48 are rewritten for the entire beam foundation in matrix form as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = a.B \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ x_1 & x_2 & x_3 & ... & x_n \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \\ \\ \\ .. \\ q_n \end{Bmatrix} \tag{10.49}$$

Equation 10.49 is simplified to

$$\{N\} = [X]\{q\} \tag{10.50}$$

where

$\{N\}$ Vector of the resulting forces and moments on the beam

$\{q\}$ Vector of contact pressures

$[X]$ Geometric matrix of the beam with coordinate *x*

Substituting Eq. 10.46 into Eq. 10.50 gives the following linear system of equations

$$\{N\} = [X]\{k_s\}[X]^T\ \{\Delta\} \tag{10.51}$$

Solving the system of linear equations Eq. 10.45, gives s_o and tan α. Substituting the s_o and tan α in Eq. 10.46 to find the *n* unknown contact pressures. Then, substituting also the s_o and tan α in Eq. 10.45 to find the *n* settlements.

10.6.2 Case of a uniform settlement ($e_x = 0$)

For a beam with a centric load (Figure 10.8), the settlement will be uniform ($s_i = s_o$) and the beam will not rotate (α = 0). Therefore, the unknowns of the problem are reduced to *n* contact pressures q_i and the rigid body translation s_o. The derivation of the uniform settlement for the rigid beam can be carried out by equating the settlement s_i by a uniform translation s_o at all elements on the beam.

In case of a beam with a centric load, Eq. 10.52 may be written as:

$$\{q\} = [k_s]\{s_o\} \tag{10.52}$$

where s_o is the uniform settlement of the soil at all elements under the beam.

Expanding Eq. 10.52 for all elements and equating all settlements by uniform rigid body translation s_o, yields to the contact forces as a function in terms $k_{i,j}$ of the matrix [*k*] as follows:

$$\left.\begin{array}{lllllllll} q_1 & = & k_{1,1}\, s_o & + & k_{1,2}\, s_o & + & k_{1,3}\, s_o & + \; ... \; + & k_{1,n}\, s_o \\ q_2 & = & k_{2,1}\, s_o & + & k_{2,2}\, s_o & + & k_{2,3}\, s_o & + \; ... \; + & k_{2,n}\, s_o \\ q_3 & = & k_{3,1}\, s_o & + & k_{3,2}\, s_o & + & k_{3,3}\, s_o & + \; ... \; + & k_{3,n}\, s_o \\ & & & & & & & & \\ q_n & = & k_{n,1}\, s_o & + & k_{n,2}\, s_o & + & k_{n,3}\, s_o & + \; ... \; + & k_{n,n}\, s_o \end{array}\right\} \tag{10.53}$$

Carrying out the summation of all contact pressures in Eq. 10.53, leads to:

$$\sum_{i=1}^{n} q_i = s_o \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j} \tag{10.54}$$

Replacing the sum of all contact pressures in Eq. 10.54 by the resultant *N/aB*, gives rigid body translation s_o, which equals to the settlement s_i at all nodes, is obtained from:

$$s_o = \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j}} = \frac{N}{aB \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j}} \tag{10.55}$$

Substituting the uniform rigid body translation s_o into Eq. 10.53, gives the *n* unknown contact pressures q_k by

$$q_k = \frac{N \sum_{j=1}^{n} k_{k,j}}{aB \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j}} \tag{10.56}$$

It should be noticed that Eq. 10.55 is analogous to Eq. 10.3 for *Winkler*'s model.

$$q_o = k_s s_o \tag{10.57}$$

where $k_s = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j}$ and $q_o = N / aB$

Therefore, the summation of terms $k_{i,j}$ may be used to determine the modulus of subgrade reaction k_s.

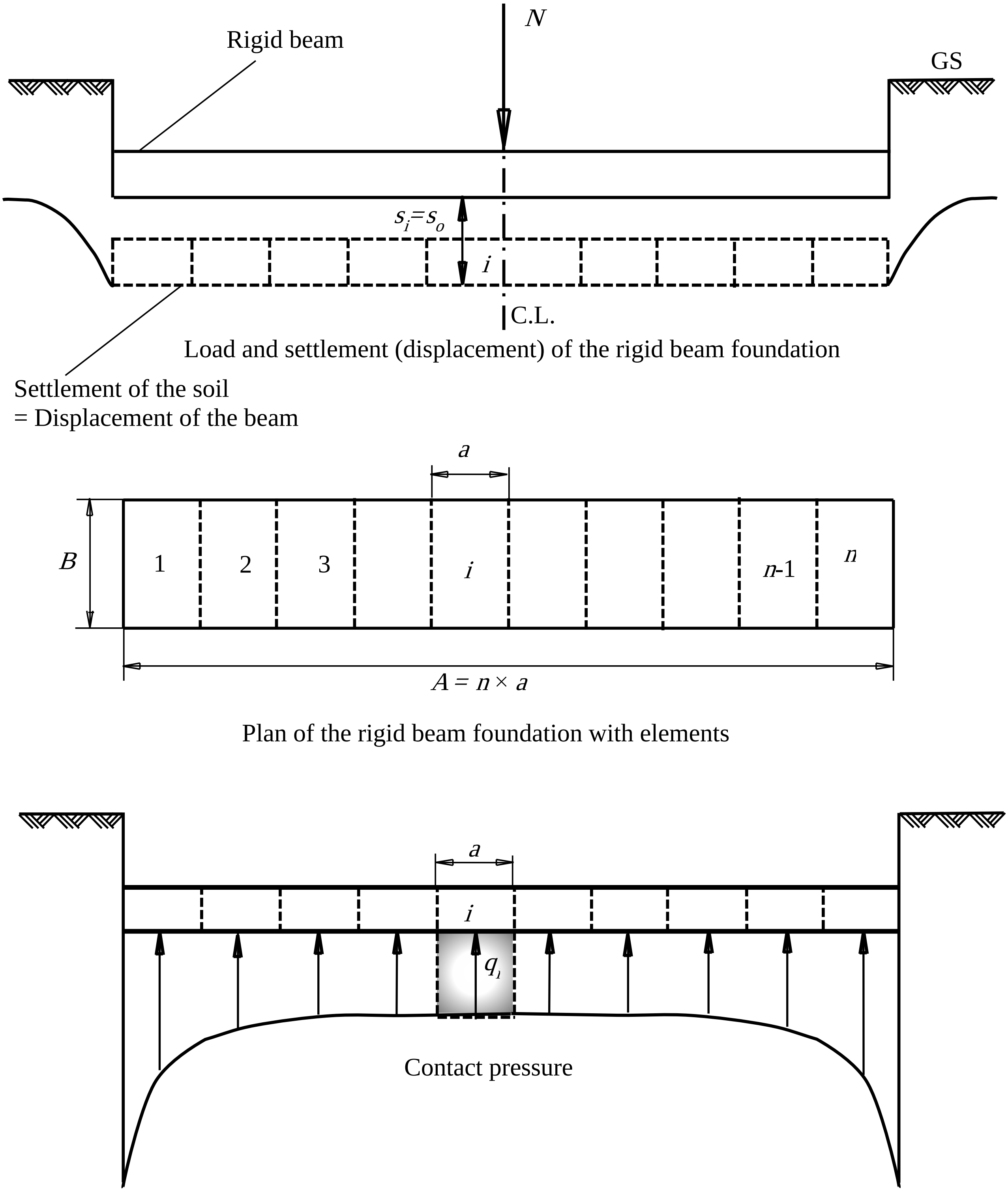


Figure 10.8 Rigid Beam Foundation by *Kany* 1972 (case of a uniform settlement)

10.7 Flexible beam foundation

In addition to the possibility for analyzing elastic and rigid beams by *GEO Tools*, the algorithm described before can be also used to calculate the settlement of a flexible beam foundation. The contact pressure and the settlements due to loads (uniform loads and concentrated loads) on the base area can be determined using the soil properties. In this case, the stiffness of the beam is not taken into account.

If the beam foundation is perfectly flexible (such as a strip of an embankment), the contact pressures will be equal to the applied distributed loads on the beam foundation, Figure 10.9.

For the set of grid points of the beam foundation, the soil settlements are

$$\left.\begin{array}{lllllllllll} s_1 & = & c_0\,q_1 & + & c_1\,q_2 & + & c_2\,q_3 & + & \ldots & + & c_{n-1}\,q_n \\ s_2 & = & c_1\,q_1 & + & c_0\,q_2 & + & c_1\,q_3 & + & \ldots & + & c_{n-2}\,q_n \\ s_3 & = & c_2\,q_1 & + & c_1\,q_2 & + & c_0\,q_3 & + & \ldots & + & c_{n-3}\,q_n \\ & & & & & & & & & & \\ s_n & = & c_{n-1}\,q_1 & + & c_{n-2}\,q_2 & + & c_{n-3}\,q_3 & + & \ldots & + & c_0\,q_n \end{array}\right\} \tag{10.58}$$

In general:

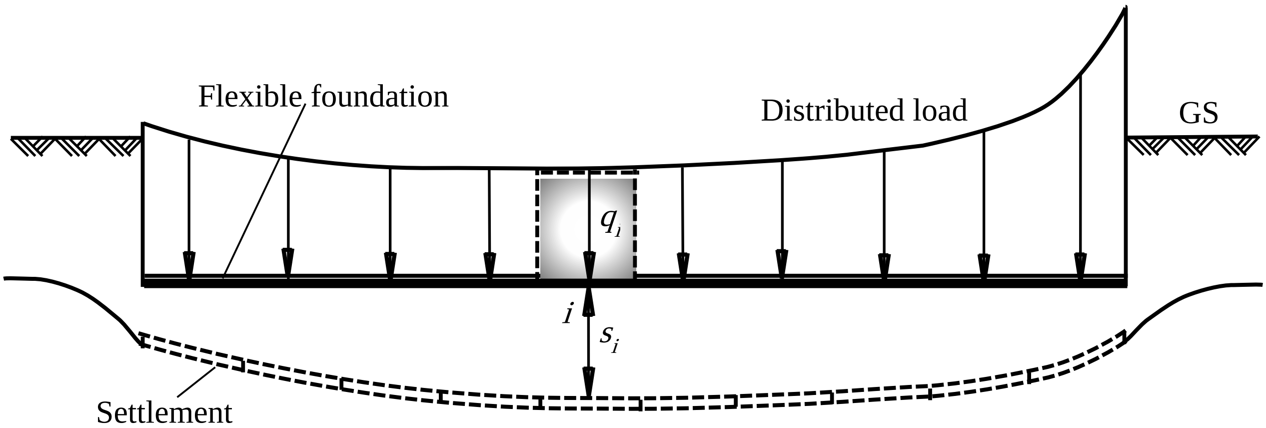
$$s_i = \sum_{j=1}^{i} c_{i-j}\,q_j + \sum_{j=i+1}^{n} c_{j-i}\,q_j \tag{10.59}$$

where c_{i-j} is the flexibility coefficient of an element i due to a unit load q_j at element j, [m^3/kN].

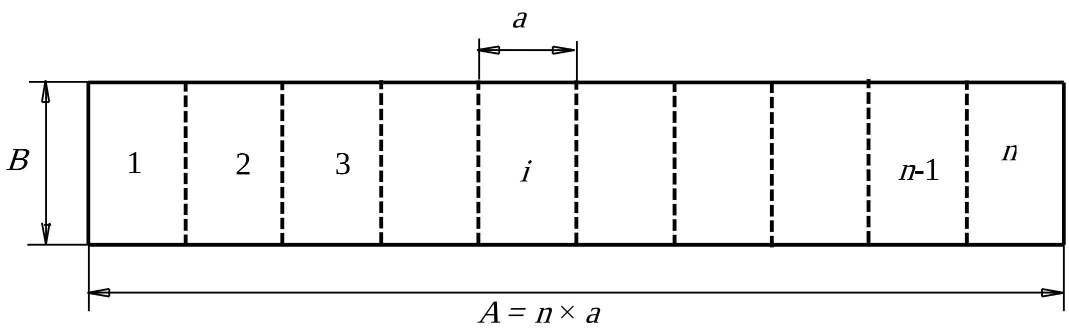
Considering the entire beam foundation, Eq. 10.58 is rewritten in matrix form as:

$$\{s\} = [c]\{q\} \tag{10.60}$$

where
$\{s\}$ Vector of settlements
$\{q\}$ Vector of contact pressures (applied distributed loads)
$[c]$ Flexibility matrix



Load and settlement of the flexible beam foundation



Plan of the flexible beam foundation with elements

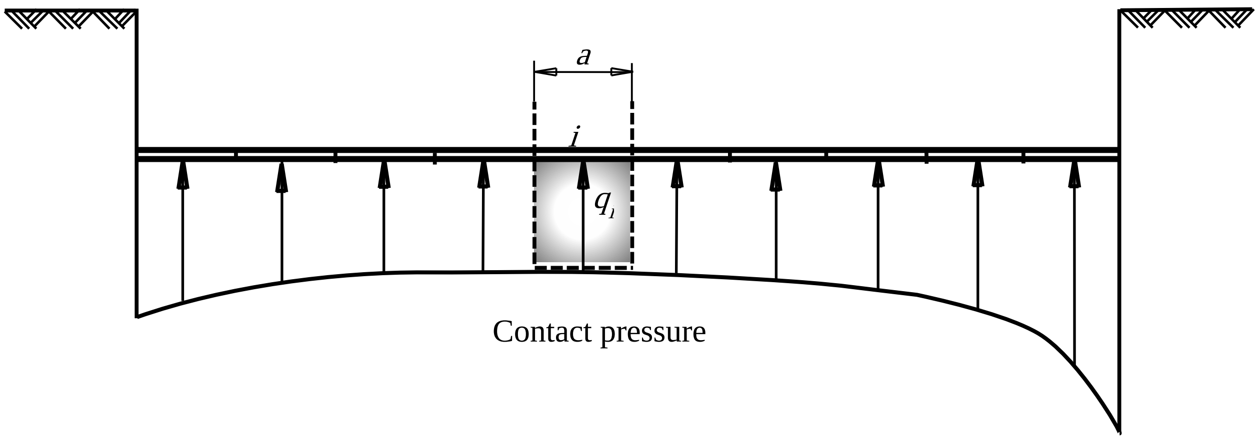


Figure 10.9 Flexible Beam Foundation

10.8 Beam foundation rigidity

The rigidity of the beam foundation depends on the ratio between the rigidity of the beam and the soil. Based on great number of comparative computations for Continuum model and *Winkler's* model, *Graßhoff* (1987) proposed various degrees of system rigidity between foundation and the soil until case of practical rigidity.

The system rigidity K_{st} for Continuum model is expressed by

$$K_{st} = \frac{E_b}{E_s}\left(\frac{d}{l}\right)^3 \tag{10.61}$$

while the system rigidity K_b for *Winkler's* model is expressed by

$$K_b = \frac{E_b}{kl}\left(\frac{d}{l}\right)^3 \tag{10.62}$$

where

E_b Modulus of Elasticity of the beam material [kN/m^2]
E_s Modulus of Compressibility of the soil [kN/m^2]
k Modulus of Subgrade Reaction of the soil [kN/m^3]
d Foundation thickness [m]
l Foundation length [m].

In which for modulus of Continuum model, $K_{st} = 1$ indicates rigid foundation and $K_{st} = 0.01$ indicates flexible foundation, while for *Winkler's* model, $K_b = 0.2$ indicates rigid foundation and $K_b = 0.002$ indicates flexible foundation.

10.9 Influences on results

10.9.1 Influence of external foundations

In many practical cases, it becomes important to assess the behavior of a foundation due to its interaction with another structural foundation or additional external loading. In this case, the settlement s_i of the beam in element i is replaced by $s_{io}+s_{i.A}$, where $s_{i.A}$ is the additional settlement due to external foundation.

Due to the external foundation, the settlement s_i at the center of element i can be expressed by

$$s_i = s_{io} + s_{i.A} \tag{10.63}$$

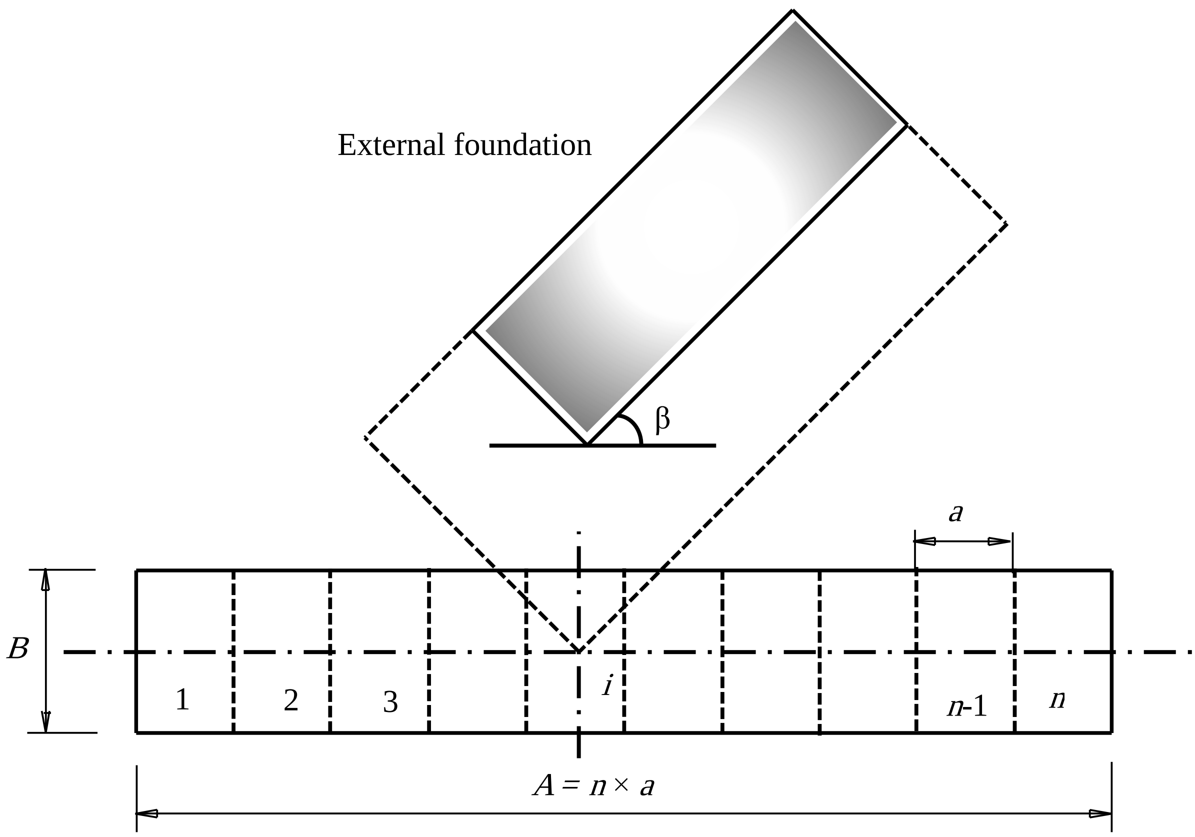
where

s_{io} Settlement at the center of element i due to the load acting upon the foundation [m]

$s_{i.A}$ Additional settlement at the center of element i due to the external foundation [m]

Due to the influences of external settlements, the right hand side R_i of Eqns 10.11 and 10.32 becomes:

$$R_i = \left(u_i M^{(l)}_{i-1} + v_i M^{(l)}_i + w_i M^{(l)}_{i+1}\right)\frac{a^2}{6E\,I_i} + s_{i-1.A} - 2\,s_{i.A} + s_{i+1.A} \tag{10.64}$$



Plan of the beam foundation under application with elements

Figure 10.10 Beam foundation under application with external foundation

10.9.2 Influence of temperature change

Sometimes, a temperature difference ΔT occurs between the upper and lower surface of the beam foundation. An example for this case is when a fire oven or boiler is constructed directly on the beam foundation in an industry structure. In this case, the settlement s_i of the beam at the center of element *i* is replaced by $s_{io}+s_{i.A}+\Delta_i$, where Δ_i is the deformation due to temperature change.

Due to the temperature change, the settlement s_i at the center of element *i* can be expressed by

$$s_i = s_{io} + s_{i.A} + \Delta_i \tag{10.65}$$

By assuming the warped surface as part of a cylinder, it can be proven from geometry, Figure 10.11, that:

$$\Delta_i = \frac{\alpha_T . \Delta T . r_i^2}{2d} \tag{10.66}$$

where

Δ_i Amount of curvature at element *i* [m]

α_T Coefficient of thermal expansion of concrete = 5×10^{-6} [1/°c]

r_i Distance from element *i* to the center of the beam where curling is zero [m]

d Thickness of the beam [m]

ΔT Temperature differential between the upper and lower surface of beam [°c], $\Delta T = T_o - T_u$

T_o Temperature at the upper surface of the beam [°c]

T_u Temperature at the lower surface of the beam [°c].

Positive deflection when the beam warps down with a temperature at the top bigger than that at the bottom.

Due to the temperature change, the right hand side R_i of Eqns 10.11 and 10.32 becomes:

$$R_i = \left(u_i M^{(l)}_{i-1} + v_i M^{(l)}_i + w_i M^{(l)}_{i+1}\right)\frac{a^2}{6E\, I_i} + s_{i-1.A} - 2\, s_{i.A} + s_{i+1.A} - \frac{\alpha_T . \Delta T . a^2}{d_i} \tag{10.67}$$

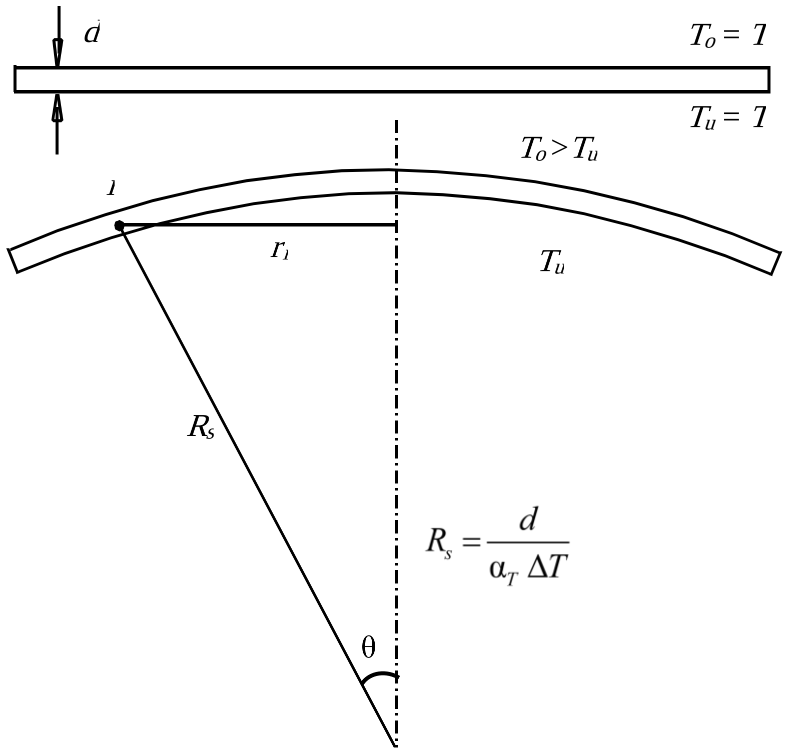


Figure 10.11 Influence of temperature change on the beam

10.9.3 Influence of groundwater pressure

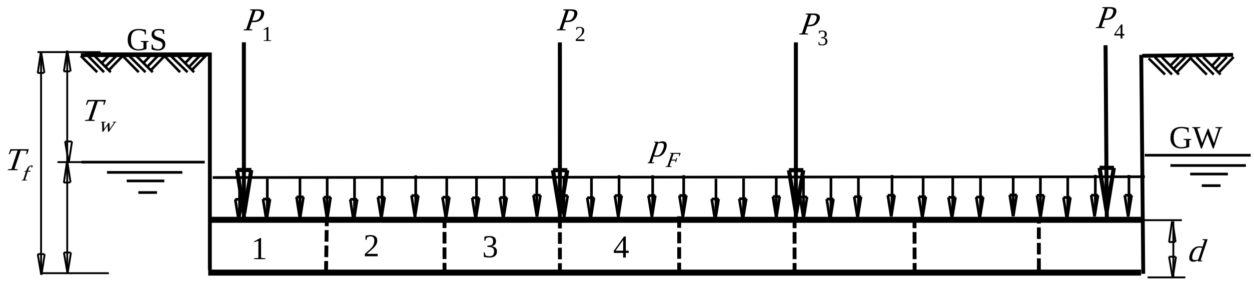
If the water table is located above the beam foundation, the beam foundation will be exposed to an additional negative pressure q_w due to the effect of groundwater. This can be taken into account in the settlement calculation. In this case, an additional negative uniform load - q_w on the base beam foundation is added to the applied uniform load on the beam.

Figure 10.12 shows an example for a beam subjected to a uniformly distributed loading p_f equal to the weight of the beam itself minus the uplift:

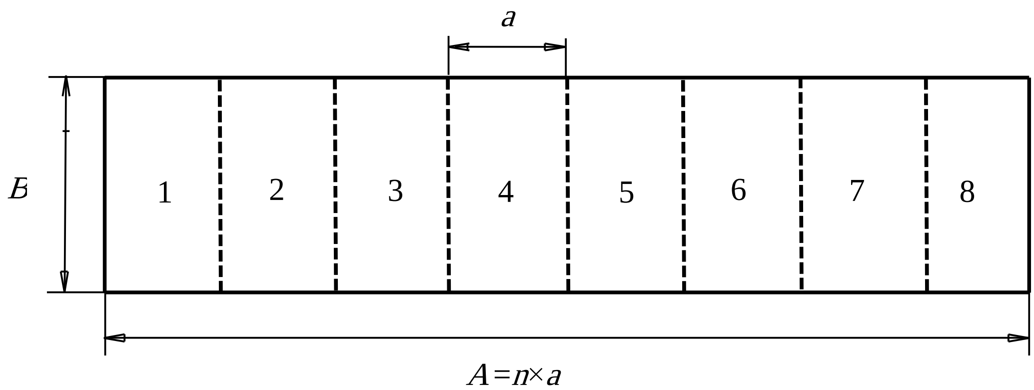
Own weight of the raft	$w_o = \gamma_b \times d$
Up lift pressure	$q_w = \gamma_w \times (T_f - Tw)$
Total	$p_f = w_o - q_w$

where

d	Thickness of beam [m]
Tw	Groundwater depth under the ground surface [m]
Tf	Foundation depth under the ground surface [m]
γ_b	Unit weight of the beam material [kN/ m^3]
γ_w	Unit weight of the water [kN/ m^3].



Beam foundation with loads



Plan of beam foundation with elements

Figure 10.12 Influence of groundwater pressure on the beam

10.10 Formulation of flexibility coefficients

First, n settlements s_i due to average soil pressure qo are calculated using the defined system of loading and subsoil data. In which, n is number of elements of the beam. Then from settlements s_i, flexibility coefficients c_i are calculated for n elements, Eq. 10.67.

$$c_i = \frac{s_i}{qo} \tag{10.68}$$

Finally, settlement differences C_i are calculated from c_i, Eq. 10.68. The settlement differences are used as input data for setting up the linear system of equations.

$$C_i = c_{i-1} - 2c_i + c_{i+1} \tag{10.69}$$

When calculating the settlements s_i, the characteristic point P_k is used. According to Figure 10.13, it is at the intersection of the straight line parallel to the y-axis in the section 0.74×a/2 from the middle of the element or parallel to the x-axis is at a distance of 0.74×b/2. Characteristic point is a point at a rectangular loaded area, in which the flexible settlement is identical with the rigid displacement.

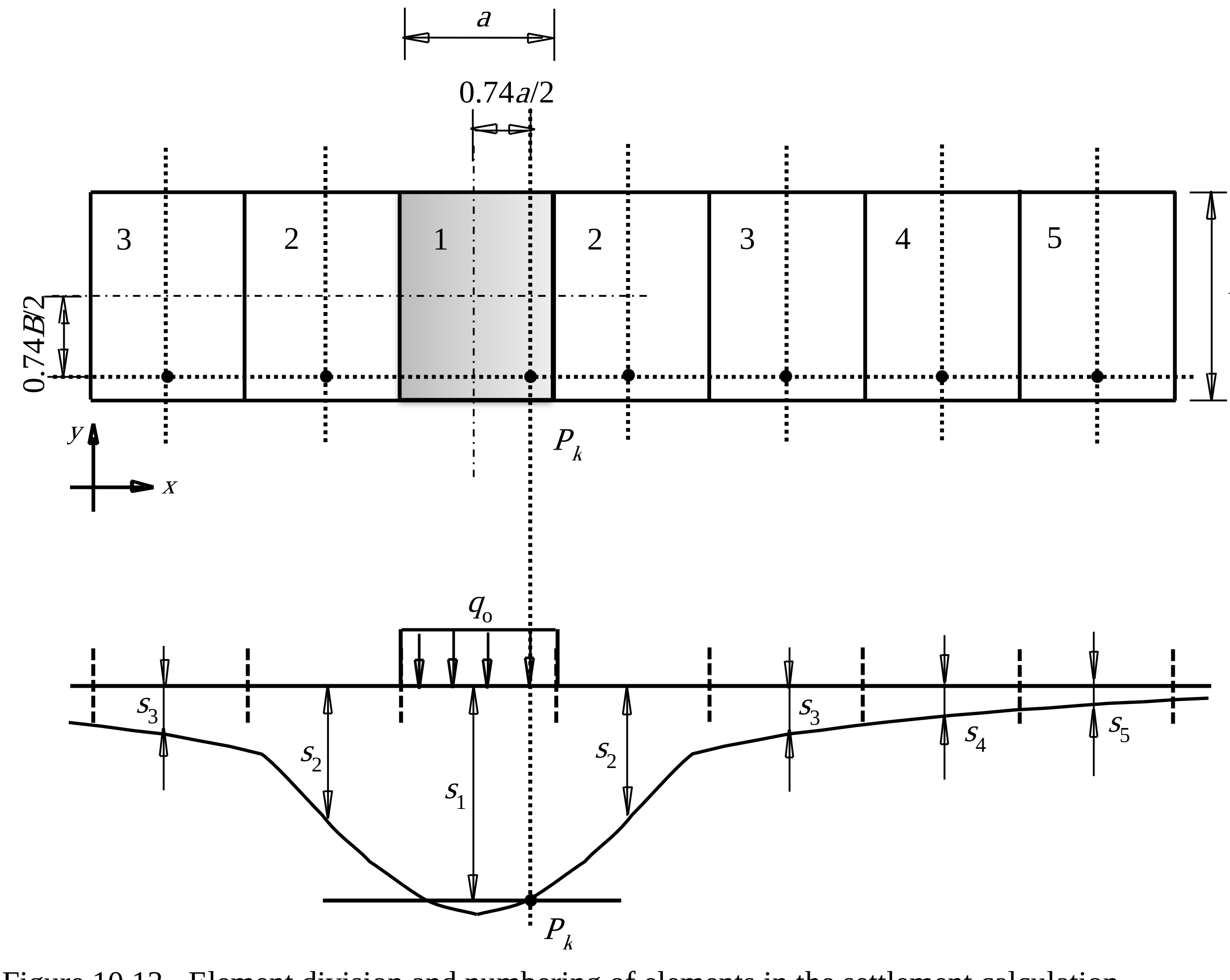


Figure 10.13 Element division and numbering of elements in the settlement calculation

In *GEO Tools*, the calculated settlements s_i, flexibility coefficients c_i and settlement differences C_i are displayed on the screen and can be printed in tables to check the results. This makes it possible to compare the calculation results with the table values after *Kany* (1974).

The following sections describe the settlement calculation.

10.11 Settlement calculation

10.11.1 Introduction

Soil medium may be considered as an isotropic elastic half-space soil medium or a layered continuum soil medium. The settlement equations of both mediums are presented in the next sections. The first one is used when a simplification for analyzing beam on elastic foundation is required. Representing the soil as a layered continuum medium is more complicated than that as an isotropic elastic half-space soil medium. *Kany* (1954) presented an extension of *Ohde*'s method (1942) to beam foundation resting on nonhomogeneous and anisotropic soil medium. It can be applied for more accurate analysis of beam on elastic foundation.

10.11.2 Settlement due to a concentrated load on half-space medium

The settlement s_i [m] at the surface outside the point of application of the concentrated load Q_j [kN] at a point j on an isotropic elastic half-space soil medium is given by, (Figure 10.14):

$$s_i = \frac{Q_j \left(1 - \nu_s^2\right)}{\pi \, E \, r_{ij}} \tag{10.70}$$

s_i Settlement under point i due to a concentrated load at point j [m]
Q_j Concentrated load at point j at the surface [kN]
r_{ij} Radial distance between points i and j [m]
E *Young's* modulus of the soil [kN/m^2]
ν_s *Poission's* ratio of the soil [-].

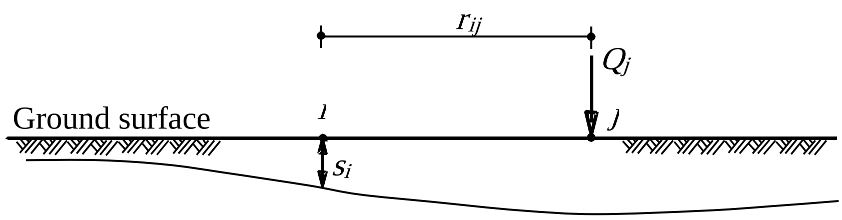


Figure 10.14 Settlement due to a concentrated load on an isotropic elastic half-space soil medium

10.11.3 Settlement due to a circular loaded area on half-space medium

The settlement s_o [m] at the surface under the center of a circular loaded area of a radius r [m] and intensity q [kN/m^2] on an isotropic elastic half-space soil medium is given by (Figure 10.15):

$$s_o = \frac{2q\, r\left(1 - \nu_s^2\right)}{E} \tag{10.71}$$

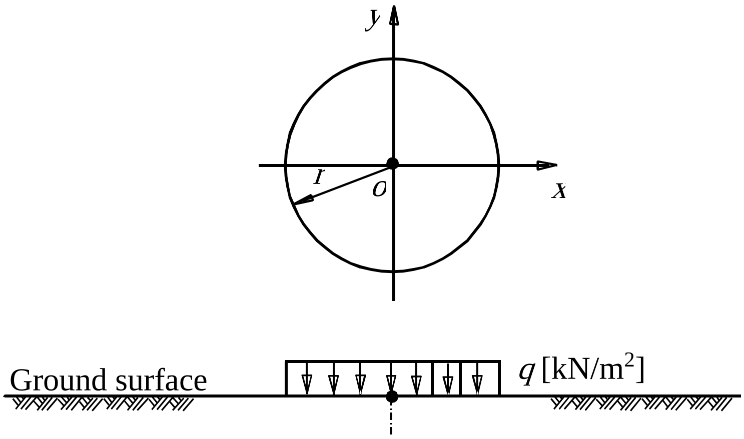


Figure 10.15 Settlement due to a circular loaded area on an isotropic elastic half-space soil medium

10.11.4 Settlement at a depth *z* due to a loaded area

According to *Steinbrenner* (1934), the settlement $s(z)$ at a depth z under the corner of the loaded area $a \times b$ and intensity q [kN/m^2] on an isotropic elastic half-space soil medium is given by (Figure 10.16):

$$s(z) = \frac{q\,(1-\nu_s^{\,2})}{2\,\pi\,E}\left(b\ln\frac{(c+a)}{(c-a)} + a\ln\frac{(c+b)}{(c-b)}\right) - \frac{q\,(1-\nu_s - 2\,\nu_s^{\,2})}{2\,\pi\,E}\left(z\tan^{-1}\frac{a\,b}{z\,c}\right) \qquad (10.72)$$

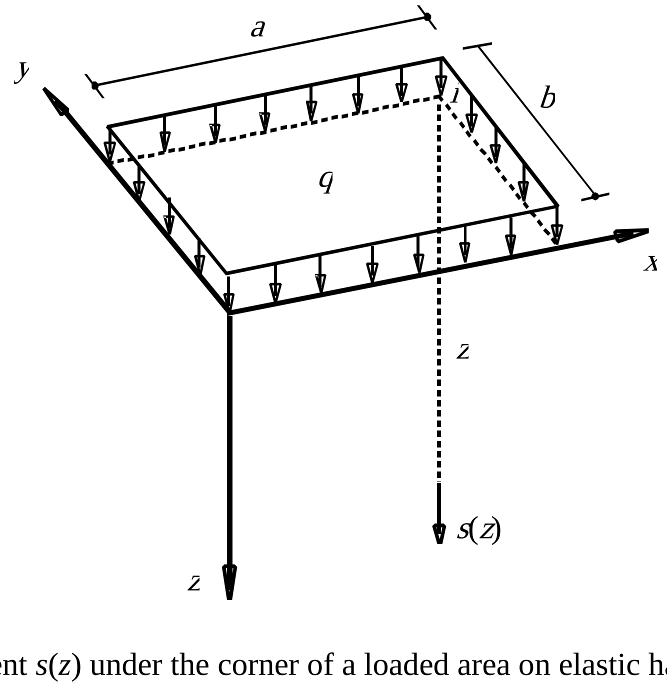


Figure 10.16 Settlement $s(z)$ under the corner of a loaded area on elastic half-space medium

10.11.5 Settlement at the surface due to a loaded area

The settlement $s(0)$ of a point at the surface under the corner of a rectangular loaded area on an isotropic elastic half-space soil medium is obtained by putting $z = 0$ in Eq. 10.72

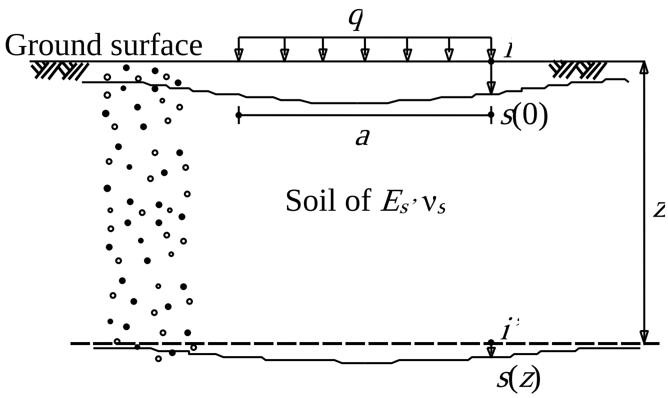
$$s(0) = \frac{q\,(1-\nu_s^{\,2})}{2\,\pi\,E}\left(b\ln\frac{(m+a)}{(m-a)} + a\ln\frac{(m+b)}{(m-b)}\right) \tag{10.73}$$

where in Eq. 10.72 and 10.73 is $c = \sqrt{a^2 + b^2 + z^2}$ and $m = \sqrt{a^2 + b^2}$

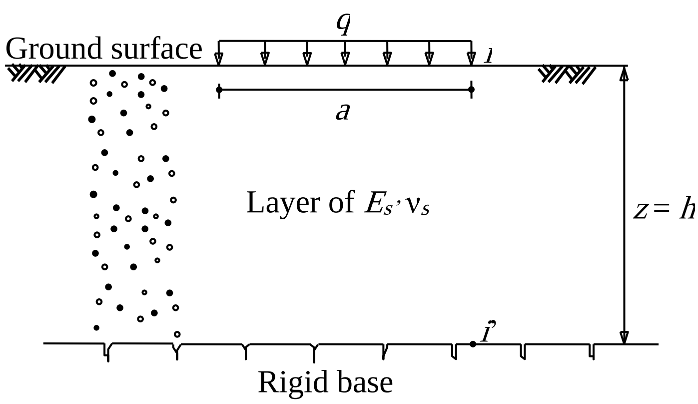
10.11.6 Settlement of a finite layer due to a loaded area

For the settlement Eqns 10.72 and 10.73 presented above, it was assumed that the soil layer extends to an infinite depth. However, if a rigid base at a depth $z = h$ underlies the soil layer, the settlement s_h of the layer can be approximately calculated as (Figure 10.17):

$$s_h = s(0) - s(z) \tag{10.74}$$



a)



b)

Figure 10.17 a) Isotropic elastic half-space soil medium

b) Elastic layer on rigid base

Subtracting Eq. 10.72 from Eq. 1.73 yields

$$s_h = \frac{q\,(1-\nu_s^{\,2})}{2\,\pi\,E}\left(b\ln\frac{(c-a)\,(m+a)}{(c+a)\,(m-a)} + a\ln\frac{(c-b)\,(m+b)}{(c+b)\,(m-b)}\right) - \frac{q\,(1-\nu_s-2\,\nu_s^{\,2})}{2\,\pi\,E}\left(z\tan^{-1}\frac{a\,b}{z\,c}\right) \quad (10.75)$$

Equation 10.75 can be simplified to

$$s_h = \frac{q}{E}\,f \quad (10.76)$$

10.11.7 Settlement of multi-layers due to a loaded area

Obviously, it can generalize this approach to consider multi-layers of soil. Each has different elastic material and thickness as shown in Figure 10.18. The vertical settlement of a layer *l* in an *n*-layered system is given by

$$s_l = q\left(\frac{f^{(l)} - f^{(l-1)}}{E^{(l)}}\right) = q\left(\frac{\Delta\,f^{(l)}}{E^{(l)}}\right) \quad (10.77)$$

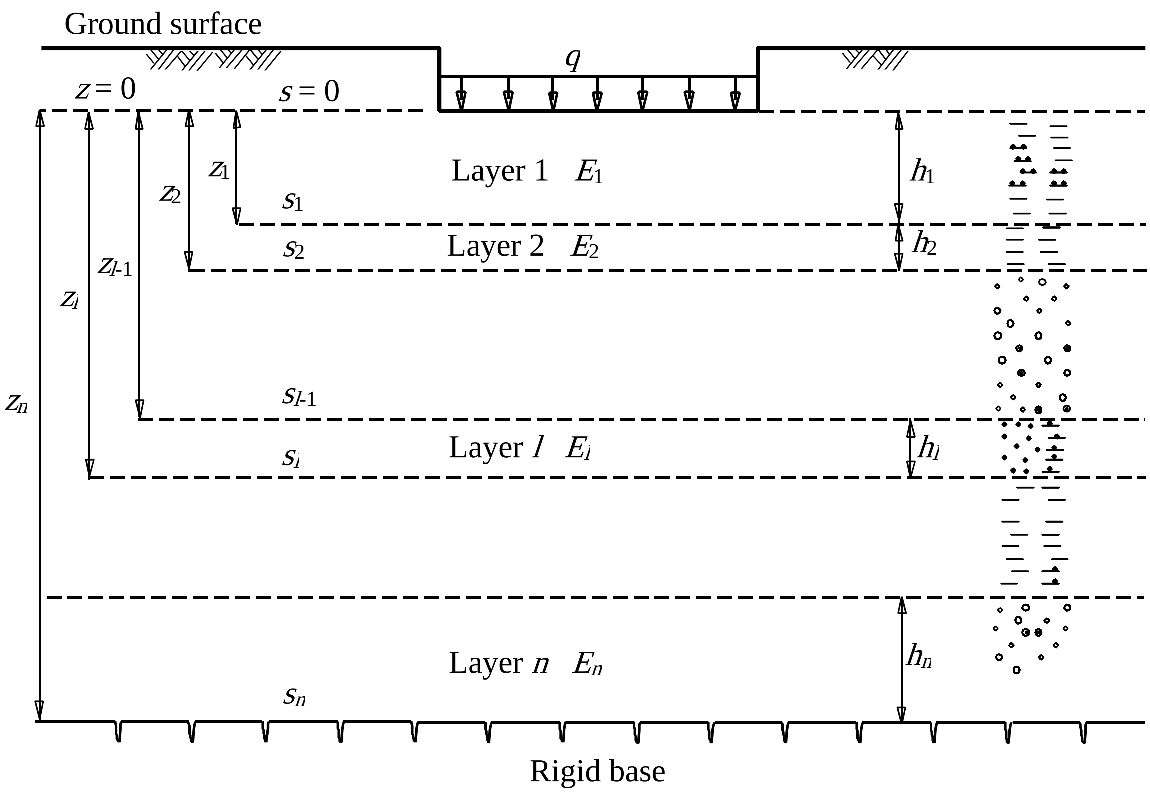


Figure 10.18 Layered system

The total settlement for *n*-layered system is

$$s = q\left(\frac{f^{(1)}}{E^{(l)}} + \sum_{l=2}^{n}\frac{\Delta f^{(l)}}{E^{(l)}}\right) \tag{10.78}$$

Considering *Poisson*'s ratio ν_s for all soil layers is constant as its value for most soil types ranges between 0.3 and 0.5.

10.11.8 Settlement at an interior point of loaded area

So far, it has been considered the settlement beneath a corner of a loaded area. To find the settlement at any other point, the principle of superposition can be used. The settlement at an interior point of the rectangular loaded area is given by the sum of the settlements at the corners of four sub-loaded areas. To determine the settlement coefficient $f^{(l)}$ for a layer l at an interior point i of the rectangular loaded area shown in Figure 10.19, the Formula of *Kany* (1974) can be applied as

$$f^{(l)} = f^{(l)}{}_1 + f^{(l)}{}_2 + f^{(l)}{}_3 + f^{(l)}{}_4$$

$$= \frac{1}{2\pi}\sum_{n=1}^{4}\left[(1-\nu_s{}^2)\left\{b_n \ln\frac{(c_n - a_n)(M + a_n)}{(c_n + a_n)(M - a_n)} + a_n \ln\frac{(c_n - b_n)(M + b_n)}{(c_n + b_n)(M - b_n)}\right\} + (1-\nu_s - 2\nu_s{}^2)\, z_l \tan^{-1}\frac{a_n\, b_n}{z_l\, c_n}\right] \tag{10.79}$$

where $c_n = \sqrt{a_n{}^2 + b_n{}^2 + z_l{}^2}$ und $M = \sqrt{a_n{}^2 + b_n{}^2}$

The value z_l means the level of the lower side of the layer l from the foundation level.

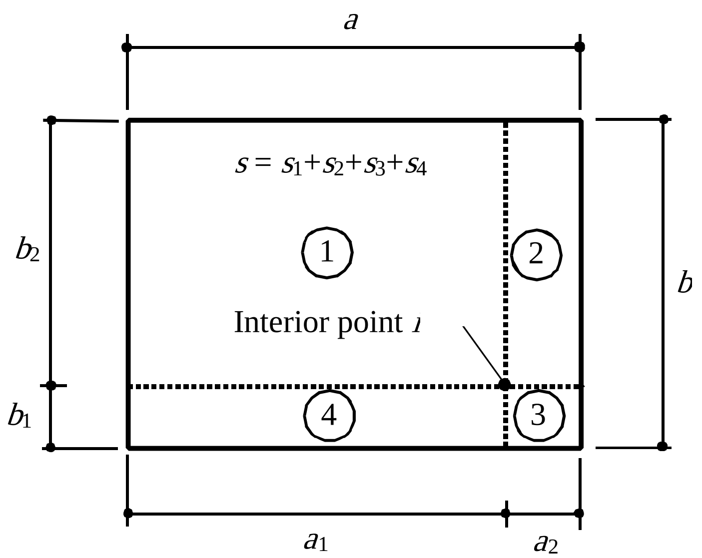


Figure 10.19 Superposition of four loaded areas to find the settlement at an interior point *i*

10.11.9 Settlement at a point outside the loaded area

Adding and subtracting corner settlements for four loaded areas can obtain the settlement of any point outside the loaded area as shown in Figure 10.20. First, the settlement s_1 as if the entire region defined by load q is determined. Then, the settlements due to the two edge loaded areas s_2 and s_3 are subtracted. Finally, the settlement s_4 is added since it has been subtracted twice in s_2 and s_3. Using the same process, the settlement coefficient $f^{(l)}$ for a layer l at an exterior point i of the rectangular loaded area shown in Figure 10.20 is given by

$$f^{(l)} = f^{(l)}{}_1 - f^{(l)}{}_2 - f^{(l)}{}_3 - f^{(l)}{}_4 \qquad (10.80)$$

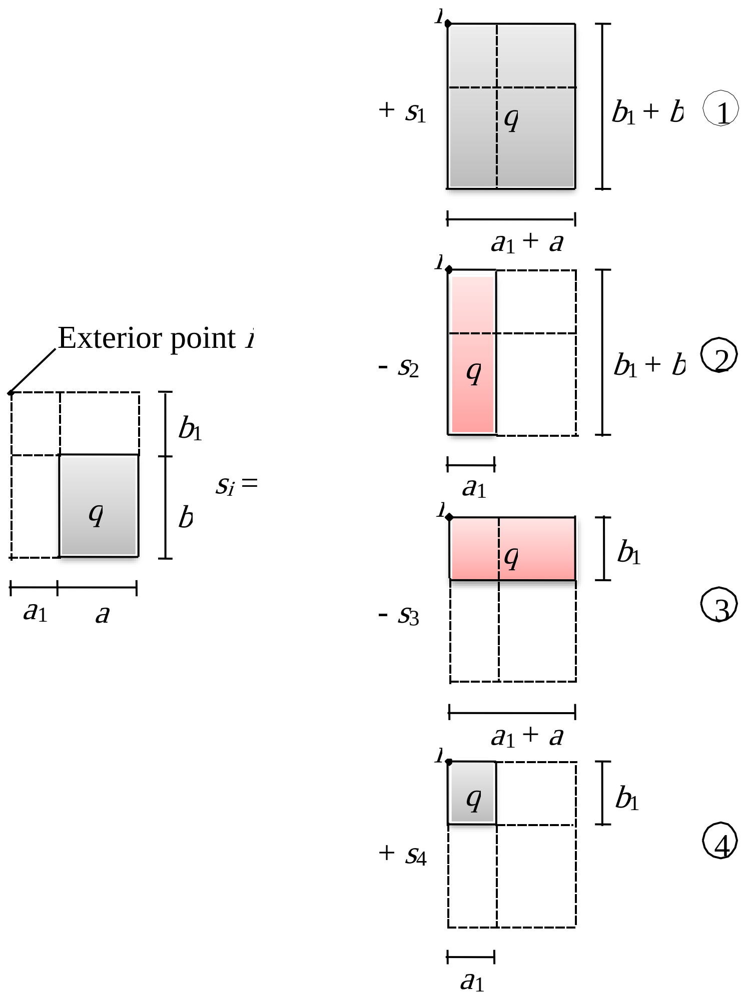


Figure 10.20 Superposition of four loaded areas to find the settlement at an exterior point i

10.12 Determination of limit depth

The assumption of the isotropic elastic half-space soil medium requires an infinite soil layer having the same compressibility under the foundation. Practically, the soil consists of many layers with different soil materials. For layered soil medium, the number of layers in a boring to be considered when determining the settlement depends on the level of the rigid surface or on the limit depth z_g where no settlement occurs. The limit depth z_g under the foundations is the level of which the stress σ_U reaches a standard ratio ξ of the initial vertical stress σ_V as indicated in Figure 10.21 and the following equation

$$\sigma_U = \zeta \ \sigma_V \tag{10.81}$$

where

$\sigma_U = \sigma_E + \sigma_D$	Stress due to the foundation load and the external foundation loads [kN/m^2]
σ_E	Stress due to the foundation load [kN/m^2]
σ_D	Stress due to the external foundation loads [kN/m^2]
$\sigma_V = \Sigma\gamma z$	Stress due to the self-weight of the soil layers [kN/m^2]
γ	Unit weight of the soil layer [kN/m^3]
z	Depth of the soil layer [m].

An examination from *Amman*/ *Breth* (1972) showed that the values ξ may be taken as ξ = 0.8, especially for reloading soil. The standard value of ξ according to DIN 4019 is ξ = 0.2.

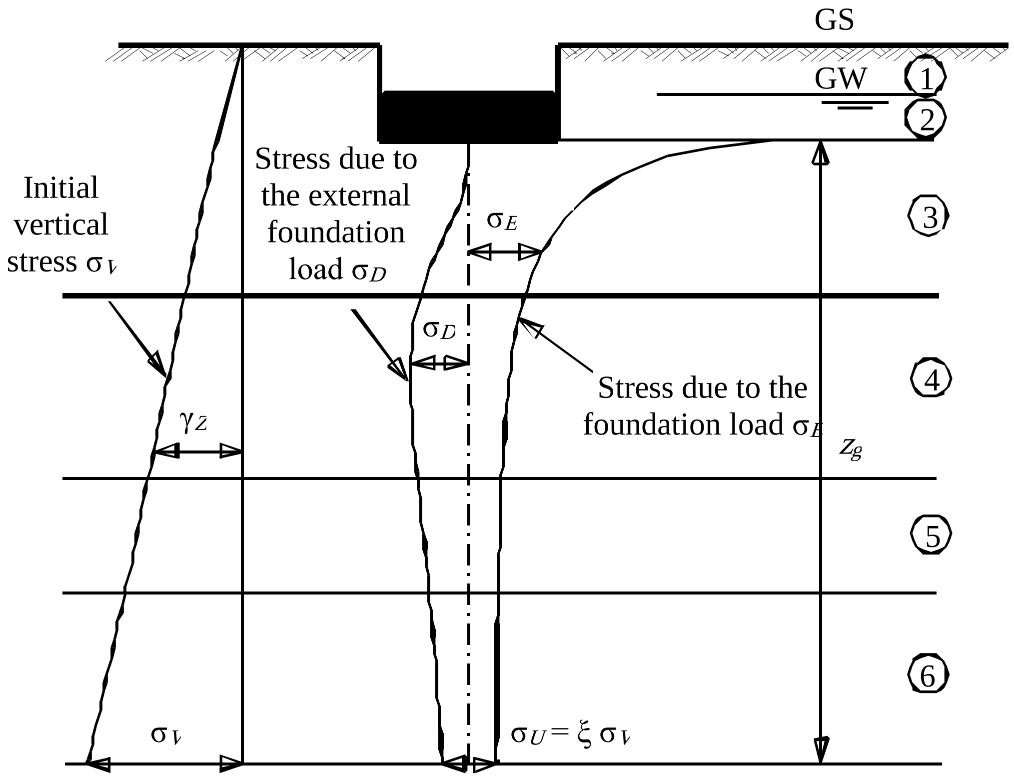


Figure 10.21 Limit depth z_g under a foundation

10.13 Bilinear soil behavior

10.13.1 General

The yielding of the subsoil is described by the modulus of compressibility (or modulus of elasticity), which can be determined from the stress settlement curve. A simplified way was supposed to improve the deformation behavior of the soil by dividing the stress settlement curve into two regions, Figure 10.22.

- In the first region, the ground will settle until reaching an overburden load *qv* according to the modulus of compressibility W_s.

- In the second region after reaching the load q_v the ground will settle more under load *q* according to the modulus of compressibility E_s until reaching the total load *qo*.

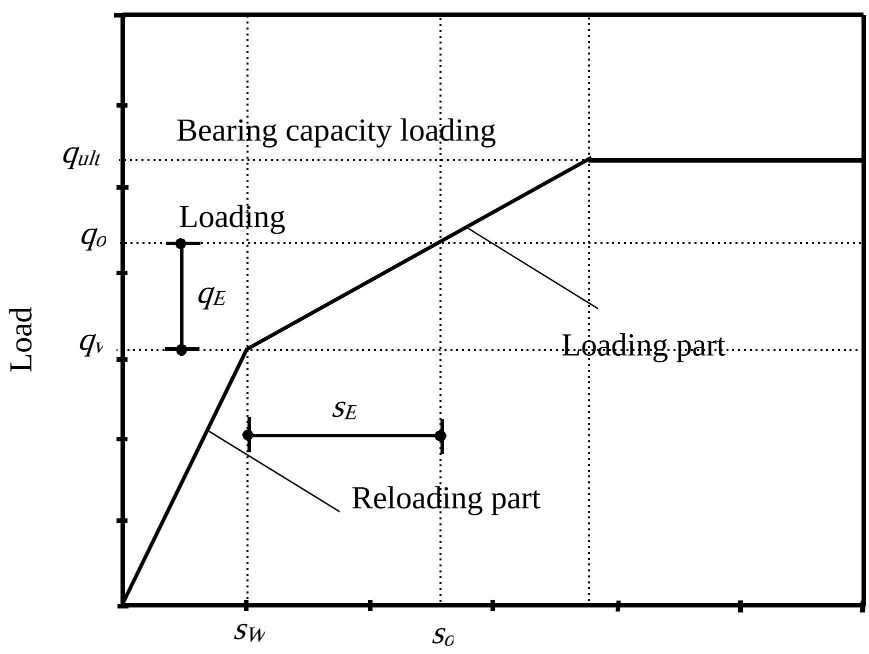


Figure 10.22 Load settlement diagram (bilinear relation)

At first, it should be carried out a primary calculation by one of the following two cases:

Case 1: ***qv<qo***

The settlement s_i of the beam foundation at the center of element *i* can be derived from two variations such that

$$s_i = s_{W\,i} + s_{E\,i} \tag{10.82}$$

where

s_{Wi} Settlement at the center of element *i* due to load from o to q_v with modulus of compressibility W_s (part of reloading) [m]

s_{Ei} Settlement at the center of element *i* due to load from q_v to q_o with modulus of compressibility E_s (part of primary loading) [m].

It can be generally said that the total contact pressure on the beam foundation is given by

$$qo = qu + qe \tag{10.83}$$

where $qu=qv$ is the overburden pressure [kN/m^2]

Case 2: ***qv>qo***
The settlement equation will be

$$s_i = s_{W\,i} \tag{10.84}$$

In this case, the contact pressure on the beam foundation is *qo,* where $qo < qv$

If one of the above two cases is not existent, an iterative solution for the settlement equation will be necessary.

The bilinear relation of the soil deformation may be taken into consideration as follows:

10.13.2 Bilinear soil behavior for elastic beam foundation (*Kany/ El Gendy* 1995)

First, n settlements s_{Ei} due to loading part qe and n settlements s_{Wi} due to reloading part qu are calculated using the defined system of loading and subsoil data. In which, n is number of elements of the beam. Then from settlements s_{Ei} and s_{Wi}, flexibility coefficients c_{ei} and c_{wi} are calculated for n elements, Eq. 10.85.

$$ce_i = \frac{s_{Ei}}{qe}$$
$$cw_i = \frac{s_{Wi}}{qu} \tag{10.85}$$

For element i, the summation equations of the settlements are given by

$$s_{Wi} = \sum_{j=1}^{i} cw_{i-j}\, qu + \sum_{j=i+1}^{n} cw_{j-i}\, qu$$
$$s_{Ei} = \sum_{j=1}^{i} ce_{i-j}\, qe + \sum_{j=i+1}^{n} ce_{j-i}\, qe \tag{10.86}$$

where

qv	Overburden pressure [kN/m^2]
$qu = qv$	Reloading contact pressure [kN/m^2]
qe	Loading contact pressure [kN/m^2]
$qo = qv + qe$	Average soil pressure [kN/m^2].

The mean modulus of subgrade reaction k_m for the whole beam is then given by

$$k_m = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{qu + qe}{s_{Wi} + s_{Ei}}\right) \tag{10.87}$$

10.13.3 Bilinear soil behavior for elastic beam foundation (*Kany* 1974)

First, *n* settlements s_{Ei} due to loading part and *n* settlements s_{Wi} due to reloading part are calculated from the defined profile and all layers below the foundation base using the defined system of loading and subsoil data. In which, *n* is number of elements of the beam. Then from settlements s_{Ei} and s_{Wi}, flexibility coefficients ce_i and cw_i are calculated for *n* elements, Eq. 10.85.

$$ce_i = \frac{s_{Ei}}{qe} \qquad cw_i = \frac{s_{Wi}}{qu} \qquad (10.88)$$

The loading contact pressure *qe* is given by

$$qe = qo - qv - qw \qquad (10.89)$$

where

qv	Overburden pressure [kN/m^2]
qu= qv	Reloading contact pressure [kN/m^2]
qe	Loading contact pressure [kN/m^2]
qw	Groundwater pressure [kN/m^2]
qo	Average soil pressure [kN/m^2].

Finally, settlement differences Ce_i and Cw_i are calculated from ce_i and cw_i values, Eq. 10.90. The settlement differences are used as input data for setting up the linear system of equations.

$$\begin{aligned} Ce_i &= ce_{i-1} - 2ce_i + ce_{i+1} \\ Cw_i &= cw_{i-1} - 2cw_i + cw_{i+1} \end{aligned} \qquad (10.90)$$

In *GEO Tools*, the calculated settlements s_{Ei} and s_{Wi}, flexibility coefficients ce_i and cw_i and settlement differences Ce_i and Cw_i are displayed on the screen and can be printed in tables to check the results. This makes it possible to compare the calculation results with the table values after *Kany* (1974).

10.13.3.1 Settlements s_i

The settlement s_i at the center of the element *i* for linear behavior is given by

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j \qquad (10.91)$$

For bilinear behavior

$$\begin{aligned} s_i &= s_{Wi} + s_{Ei} \\ &= \left(\sum_{j=1}^{i} cw_{i-j}\; qu + \sum_{j=i+1}^{n} cw_{j-i}\; qu \right) + \left(\sum_{j=1}^{i} ce_{i-j}\; qe_j + \sum_{j=i+1}^{n} ce_{j-i}\; qe_j \right) \end{aligned} \qquad (10.92)$$

10.13.3.2 Moments M_i

Clapeyron's three-moment equation for linear behavior is given by

$$- s_{i-1} + 2\, s_i - s_{i+1} = \left(u_i\, M_{i-1} + v_i\, M_i + w_i\, M_{i+1}\right) \frac{a^2}{6\, E\, I_i} \qquad (10.93)$$

For bilinear behavior

$$\left(-s_{Ei-1}+2\,s_{Ei}-s_{Ei+1}\right)+\left(-s_{Wi-1}+2\,s_{Wi}-s_{Wi+1}\right)=\left(u_i\,M_{i-1}+v_i\,M_i+w_i\,M_{i+1}\right)\frac{a^2}{6\,E\,I_i} \tag{10.94}$$

The moment M_i of external forces at the center of element *i* for linear behavior is given by

$$M_i = M_{Rl}+a^2\,B\sum_{j=1}^{i}(i-j)\,q_j-M_i^{(l)} \tag{10.95}$$

For bilinear behavior:

$$M_i = M_{Rl}+a^2\,B\sum_{j=1}^{i}(i-j)\,qe_j+a^2\,B\sum_{j=1}^{i}(i-j)\,qu-M_i^{(l)} \tag{10.96}$$

10.13.3.3 Contact pressures q_i for general case

By eliminating s_i and M_i from Equations (10.92), (10.94) and (10.96), the following equation for i= 2 to i= n-2 can be obtained:

$$\begin{aligned}&\sum_{j=2}^{i}\left(Ce_{i-j+1}+\left[(i-j)u_i+(i-j+1)v_i+(i-j+2)w_i\right]\frac{\alpha_i}{6}\right)qe_j+\left(Ce_0+w_i\frac{\alpha_i}{6}\right)qe_i+\sum_{j=1}^{n-i}Ce_j\,qe_{i+j}\\&+\sum_{j=2}^{i}\left(Cw_{i-j+1}+\left[(i-j)u_i+(i-j+1)v_i+(i-j+2)w_i\right]\frac{\alpha_i}{6}\right)qu+\left(Cw_0+w_i\frac{\alpha_i}{6}\right)q_u+\sum_{j=1}^{n-i}Cw_j\,qu\\&=\left(u_i\,M^{(l)}_{i-1}+v_i\,M^{(l)}_i+w_i\,M^{(l)}_{i+1}\right)\frac{a^2}{6E\,I_i}\end{aligned} \tag{10.97}$$

or

$$\sum_{j=2}^{i}\left(Ce_{i-j+1}+\left[(i-j)u_i+(i-j+1)v_i+(i-j+2)w_i\right]\frac{\alpha_i}{6}\right)qe_j+\left(Ce_0+w_i\frac{\alpha_i}{6}\right)qe_i+\sum_{j=1}^{n-i}Ce_j\,qe_{i+j}=R_i \tag{10.98}$$

where

$$\begin{aligned}R_i&=\left(u_i\,M^{(l)}_{i-1}+v_i\,M^{(l)}_i+w_i\,M^{(l)}_{i+1}\right)\frac{a^2}{6E\,I_i}\\&-\sum_{j=2}^{i}\left(Cw_{i-j+1}+\left[(i-j)u_i+(i-j+1)v_i+(i-j+2)w_i\right]\frac{\alpha_i}{6}\right)qu-\left(Cw_0+w_i\frac{\alpha_i}{6}\right)qu-\sum_{j=1}^{n-i}Cw_j\,qu\end{aligned} \tag{10.99}$$

The constants Ce_i and Cw_i are related to ce_i and cw_i respectively, where

$$Ce_o=2\left(ce_1-ce_0\right),\quad Ce_1=ce_0-2ce_1+ce_2,\quad Ce_2=ce_1-2ce_2+ce_3,\ldots Ce_{n-2}=ce_{n-3}-2ce_{n-2}+ce_{n-1}$$

$$Cw_o=2\left(cw_1-cw_0\right),\quad Cw_1=cw_0-2cw_1+cw_2,\quad Cw_2=cw_1-2cw_2+cw_3,\ldots Cw_{n-2}=cw_{n-3}-2cw_{n-2}+ce_{n-1}$$

Equation 10.98 can be applied at elements 2 to n-2, therefore two further equations are required to obtain the n unknown contact pressures qe_1 to qe_n. This can be done by considering the overall equilibrium of the vertical forces and moments of the beam foundation.

10.13.3.4 Equilibrium of the vertical forces:

The resultant N due to external vertical forces acting on the raft must be equal to the sum of contact forces

$$\left.\begin{aligned}&\Sigma V = 0\\&a.B\left(qe_1 + qe_2 + qe_3 + \ldots + qe_e\right) + n.a.B.qu - \Sigma P = 0\end{aligned}\right\} \qquad (10.100)$$

10.13.3.5 Equilibrium of the moments about y-axis:

Furthermore, the moments around the *y*-axis must be in equilibrium

$$\left.\begin{aligned}&\Sigma M = 0\\&\frac{(2n-1)}{2} qe_1 a^2 B + \frac{(2n-3)}{2} qe_2 a^2 B + \frac{(2n-5)}{2} qe_3 a^2 B + \ldots + \frac{1}{2} qe_n a^2 B - M_{Rl} + M_{Rr} - \Sigma M^{(l)}\\&+ \frac{n^2 . a^2 . B.qu}{2} = 0\end{aligned}\right\} \qquad (10.101)$$

Equation 10.98 to Eq. 10.101 can then be used to obtain the unknown soil contact pressures qe_n for any arbitrary external loading condition. Settlements of the soil under the beam foundation can be obtained by substituting the calculated contact pressures in Eq. 10.92. Once the contact pressures qe_i are obtained at the various sections, then the internal forces in the beam can be calculated.

10.13.4 Bilinear soil behavior for rigid beam foundation (*Kany* 1972)

For every element *i*, the summation equations of the settlements s_i are first set up with the initially unknown contact pressure q_i

$$s_i = \sum_{k=1}^{n} ce_{i,k}\ qe_k + \sum_{k=1}^{n} cw_{i,k}\ qu + s_{i.A} \Bigg\} \tag{10.102}$$

where

$ce_{i,k}$ Flexibility coefficient of element *i* due to a unit loading contact pressure $qe_k = 1$ at the element *k* [m^3/kN]

$cw_{i,k}$ Flexibility coefficient of element *i* due to a unit reloading contact pressure $qu = 1$ at the element *k* [m^3/kN]

s_i Settlement at element *i* [m]

$s_{i.A}$ Additional settlement at element *i* [m]

qe_k Loading contact pressure on element *k* [kN/m^2]

qu Reloading contact pressure [kN/m^2].

Equation (10.102) in matrix form:

$$\{s\} = [C_e]\{q_e\} + [C_w]\{q_u\} + \{s_A\} \tag{10.103}$$

$$[k_s]\{s\} = \{q_e\} + [k_s][C_w]\{q_u\} + [k_s]\{s_A\} \tag{10.104}$$

$$[k_s]\{s\} = \{q_e\} + [k_s]\{s_w\} + [k_s]\{s_A\} \tag{10.105}$$

$$[k_s]\{s\} = \{\{q_e\} + \{q_u\}\} + \{[k_s]\{s_w\} + [k_s]\{s_A\} - \{q_u\}\} \tag{10.106}$$

$$[k_s]\{s\} = \{q\} + \{P_t\} \tag{10.107}$$

$$\{q\} = [k_s]\{s\} - \{P_t\} \tag{10.108}$$

where

$\{s_A\}$ Vector of additional settlement

$\{q_e\}$ Vector of loading contact pressure

$\{q_u\}$ Vector of reloading contact pressure

$[C_e]$ Flexibility coefficient matrix for loading part

$[C_w]$ Flexibility coefficient matrix for reloading part

$[k_s] = [C_e]^{-1}$ Soil stiffness matrix for loading part

Substituting Eq. 10.108 into Eq. 10.50, gives

$$\{N\} = [X]\{q\} = [X]\{[k_s]\{s\} - \{P_t\}\} \tag{10.109}$$

$$\{N\} = [X][k_s]\{s\} - [X]\{P_t\} \tag{10.110}$$

$$\{N\} + [X]\{P_t\} = [X][k_s]\{s\} \tag{10.111}$$

Substituting Eq. 10.45 into Eq. 10.111, gives

$$\{N\}+[X]\{P_t\}=[X][k_s][X]^T\{\Delta\} \tag{10.112}$$

$$\{V\}=[E]\{\Delta\} \tag{10.113}$$

Solving the system of linear equations 10.113 to get s_o and tan α. Substituting these values in Equation 10.45, gives n unknown soil settlements s_i. Then substituting the n soil settlements in Equation 10.108, gives the n unknown contact pressures

10.14 Soil properties and parameters

10.14.1 Introduction

The elastic properties of the soil are defined in *GEO Tools* by the following two different parameters:

1. Modulus of Compressibility E_s ($1/m_v$)
2. Modulus of Elasticity E

E_s [kN/m^2] is the reciprocal value of the coefficient of volume change m_v [m^2/kN]

For each soil layer, the input data maybe are

Depth of the layer from the ground surface	z	[m]
Modulus of Compressibility for loading (constant in a layer t)	E_s	[kN/m^2]
Modulus of Compressibility for reloading (constant in a layer)	W_s	[kN/m^2]
Modulus of Elasticity for loading (constant in a layer)	E	[kN/m^2]
Modulus of Elasticity for reloading (constant in a layer)	W	[kN/m^2]
Unit weight of the soil	γ_s	[kN/m^3]
Poisson's ratio of the soil	ν_s	[-].

The following sections describe these properties of the soil. Furthermore, the soil characteristics for different soil types are listed in tables, which may be used in the primary analysis.

10.14.2 *Poisson*'s ratio ν_s

Poisson's ratio ν_s for a soil is defined as the ratio of lateral strain to longitudinal strain. It can be evaluated from the Triaxial test. *Poisson*'s ratio ν_s can be determined from at-rest earth pressure coefficient K_o as follows

$$\nu_s = \frac{K_o}{1+K_o} \tag{10.114}$$

Some typical values for *Poisson*'s ratio are shown in Table 10.1 according to *Bowles* (1977). *Poisson*'s ratio in general ranges between 0 and 0.5.

Table 10.1 Typical range of values for *Poisson*'s ratio ν_s according to *Bowles* (1977)

Type of soil	*Poisson*'s ratio ν_s [-]
Clay, saturated	0.4 - 0.5
Clay, unsaturated	0.1 - 0.3
Sandy clay	0.2 - 0.3
Silt	0.3 - 0.35
Sand, dense	0.2 - 0.4
Sand, coarse (void ratio = 0.4 - 0.7)	0.15
Sand, fine grained (void ratio = 0.4 - 0.7)	0.25
Rock	0.1 - 0.4

10.14.3 Moduli of compressibility E_s and W_s

The equations derived in the previous section for calculation of flexibility coefficients require either the moduli of compressibility for loading E_s and reloading W_s or moduli of elasticity for loading E and reloading W for the soil. The yielding of the soil is described by these elastic moduli. The moduli of compressibility E_s and W_s can be determined from the stress-strain curve through a confined compression test (for example Odometer test) as shown in Figure 10.23. In this case, the deformation will occur in the vertical direction only. Therefore, if the moduli of compressibility E_s and W_s are determined from a confined compression test, *Poisson*'s ratio will be taken $\nu_s = 0.0$. If the other moduli of elasticity E and W are used in the equations derived in the previous section, *Poisson*'s ratio will be taken to be $\nu_s \neq 0$. In general, *Poisson*'s ratio ranges in the limits $0 < \nu_s < 0.5$.

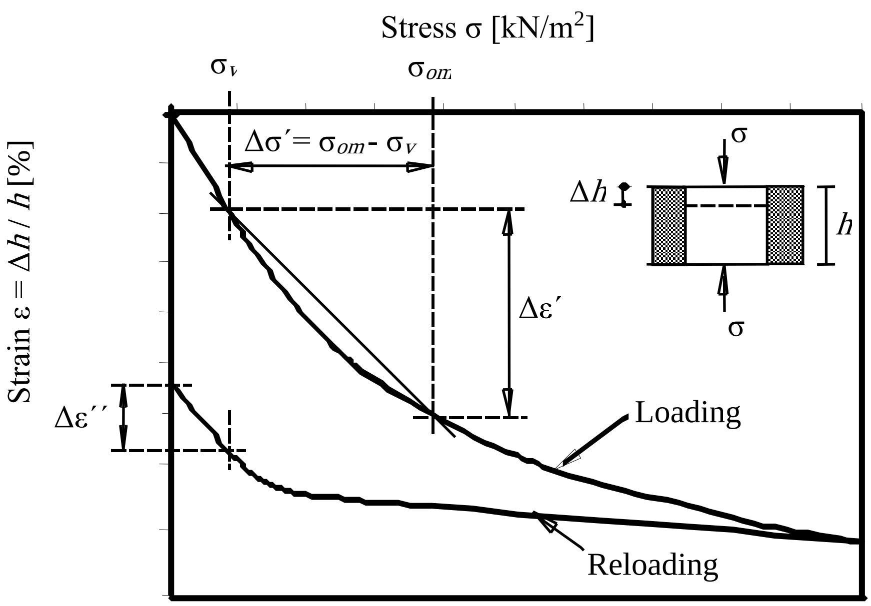


Figure 10.23 Stress-strain diagram from confined compression test (Oedometer test)

The modulus of compressibility E_s [kN/m²] (or W_s [kN/m²]) is defined as the ratio of the increase in stress $\Delta\sigma$ to decrease in strain $\Delta\varepsilon$ as (Figure 10.23)

$$\left.\begin{aligned} E_s &= \frac{\Delta\sigma'}{\Delta\varepsilon'} = \frac{\sigma_{om} - \sigma_v}{\Delta\varepsilon'} \\ W_s &= \frac{\Delta\sigma''}{\Delta\varepsilon''} = \frac{\sigma_v}{\Delta\varepsilon''} \end{aligned}\right\} \qquad (10.115)$$

where

$\Delta\sigma'$	Increase in stress from σ_v to σ_{om}	[kN/m²]
σ_v	Stress equal to overburden pressure	[kN/m²]
σ_{om}	Stress equal to expected average stress on the soil	[kN/m²]
$\Delta\varepsilon'$	Decrease in strain due to stress from σ_v to σ_{om}	[-]
$\Delta\sigma''$	Increase in stress due to reloading	[kN/m²]
$\Delta\varepsilon''$	Decrease in strain due to reloading	[-].

The moduli of compressibility may be expressed in terms of either void ratio or specimen thickness. For an increase in effective stress $\Delta\sigma$ to decrease in void ratio Δe, the moduli of compressibility E_s [kN/m^2] and W_s [kN/m^2] are then expressed as

$$\left.\begin{aligned} E_s &= \frac{1}{m'_v} = \frac{\Delta\sigma'\,(1+e'_o)}{\Delta e'} \\ W_s &= \frac{1}{m''_v} = \frac{\Delta\sigma''\,(1+e''_o)}{\Delta e''} \end{aligned}\right\} \tag{10.116}$$

where

m'_v	Coefficient of volume change for loading	[m^2/kN]
m''_v	Coefficient of volume change for reloading	[m^2/kN]
e'_o	Initial void ratio for loading	[-]
e''_o	Initial void ratio for reloading	[-]
$\Delta e'$	Decrease in void ratio due to loading	[-]
$\Delta e''$	Decrease in void ratio due to reloading	[-].

The values of E_s and W_s for a particular soil are not constant but depend on the stress range over which they are calculated. Therefore, for linear analysis it is recommended to determine the modulus of compressibility for loading E_s at the stress range from σ_v to σ_{om}, while that for reloading W_s for a stress increment equal to the overburden pressure σ_v. On the other hand, since the modulus of compressibility increases with the depth of the soil, for more accurate analysis the modulus of compressibility may be taken increasing linearly with depth. Also, according to *Kany* (1976) the moduli of compressibility E_s and W_s may be taken depending on the stress on soil. In these two cases, the moduli of compressibility E_s and W_s can be defined in the analysis for several sub-layers instead of one layer of constants E_s and W_s.

As a rule, before the analysis the soil properties are defined through the tests of soil mechanics, particularly the moduli of compressibility E_s and W_s. For precalculations Table 10.2 for specification of the modulus of compressibility E_s can also be used.

According to *Kany* (1974), the values of W_s range between 3 to 10 times of E_s. From experience, the modulus of compressibility W_s for reloading can be taken 1.5 to 5 times as the modulus of compressibility E_s for loading.

For geologically strongly preloaded soil, the calculation is often carried out only with the modulus of compressibility for reloading W_s. In this case, the same values are defined for E_s and W_s.

Matching with the reality, satisfactory calculations of the settlements are to be expected only if the soil properties are determined exactly from the soil mechanical laboratory, field tests or back calculation of settlement measurements.

Table 10.2 shows mean moduli of compressibility E_s for various types of soil according to EAU (1990).

Table 10.2 Mean moduli of compressibility E_s for various types of soil

Type of soil	Modulus of compressibility E_s [kN/m^2]
Non-cohesive soil	
Sand, loose, round	20000 - 50000
Sand, loose, angular	40000 - 80000
Sand, medium dense, round	50000 - 100000
Sand, medium dense, angular	80000 - 150000
Gravel without sand	100000 - 200000
Coarse gravel, sharp edge	150000 - 300000
Cohesive soil	
Clay, semi-firm	5000 - 10000
Clay, stiff	2500 - 5000
Clay, soft	1000 - 2500
Boulder clay, solid	30000 - 100000
Loam, semi-firm	5000 - 20000
Loam, soft	4000 - 8000
Silt	3000 - 10000

10.14.4 Moduli of elasticity *E* and *W*

The equations derived in the previous section to determine the flexibility coefficients are used with moduli of elasticity E and W for unconfined lateral strain with *Poisson*'s ratio $\nu_s \neq 0$. It must be pointed out that, when defining *Poisson*'s ratio by $\nu_s = 0$ (limit case), the moduli of compressibility E_s and W_s for confined lateral strain (for example from Odometer test) also can be used.

The modulus of elasticity is often determined from an unconfined Triaxial compression test, Figure 10.24. Plate loading tests may also be used to determine the in situ modulus of elasticity of the soil as elastic and isotropic.

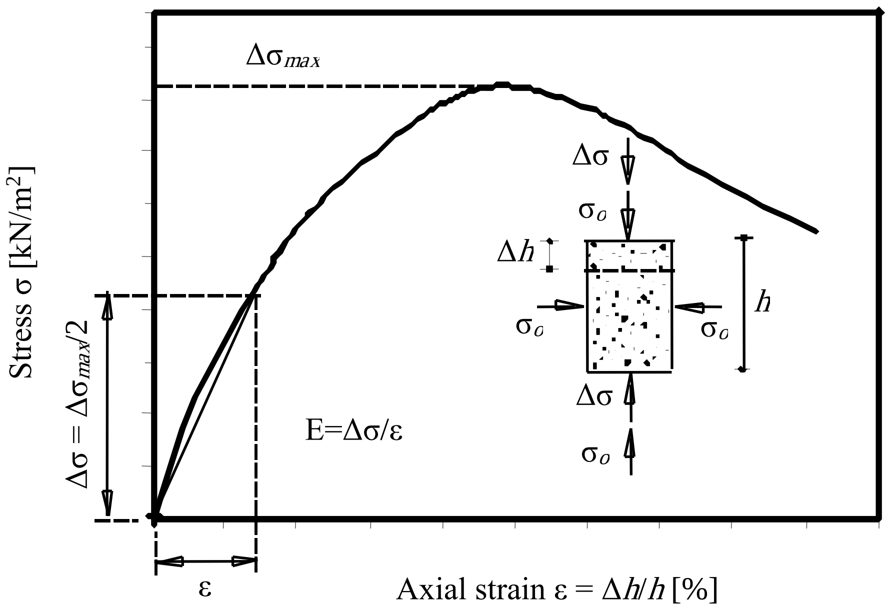


Figure 10.24 Modulus of elasticity *E* from Triaxial test

It is possible to obtain an expression for the moduli of elasticity E and W in terms of moduli of compressibility E_s, W_s and *Poisson*'s ratio ν_s for the soil as

$$\left.\begin{aligned} E &= E_s \frac{1-\nu_s-2\nu_s^{\,2}}{1-\nu_s} \\ W &= W_s \frac{1-\nu_s-2\nu_s^{\,2}}{1-\nu_s} \end{aligned}\right\} \qquad (10.117)$$

The above equation shows that:

- In the limit case $\nu_s = 0$ (deformation without lateral strain), the values of E and E_s (also W and W_s) are equal

- In the other limit case $\nu_s = 0.5$ (deformation with constant volume), the moduli of elasticity will be $E = 0 \times E_s$ and $W = 0 \times W_s$. In this case, only the immediate settlement (lateral deformation with constant volume) can be determined.

Table 10.3 shows some typical values of modulus of elasticity according to *Bowles* (1977).

Table 10.3 Typical range of moduli of elasticity *E* for selected soils

Type of soil	Modulus of elasticity E [kN/m^2]
Very soft clay	3000 - 3000
Soft clay	2000 - 4000
Medium clay	4500 - 9000
Hard clay	7000 - 20000
Sandy clay	30000 - 42500
Silt	2000 - 20000
Silty sand	5000 - 20000
Loose sand	10000 - 25000
Dense sand	50000 - 100000
Dense sand and gravel	80000 - 200000
Loose sand and gravel	50000 - 140000
Shale	140000 - 1400000

10.14.5 Modulus of subgrade reaction k_s

It is important to say that the modulus of subgrade reaction k_s is not a soil constant, but it can be related to the elastic parameters E_s and ν_s of the soil.

It may be determined from in situ plate loading test. This test is generally performed using a circular steel plate (30 in diameter) thick enough so that the bottom plate will settle uniformly under a vertical load. The modulus of subgrade reaction k_s [kN/m^3] is defined as the ratio between the soil pressure q [kN/m^2] and corresponding settlement s [m] through the following equation

$$k_s = \frac{q}{s} \tag{10.118}$$

In practice, the plate would not stress the same soil strata as the full size foundation. Therefore, the result from a plate-loading test may give quite misleading results if the proposed foundation is large. The soft layer of soil in Figure 10.25 is unaffected by the plate loading test but would be considerably stressed by the foundation. Therefore, it is recommended to evaluate the modulus k_s from the elastic parameters E_s and ν_s of the soil.

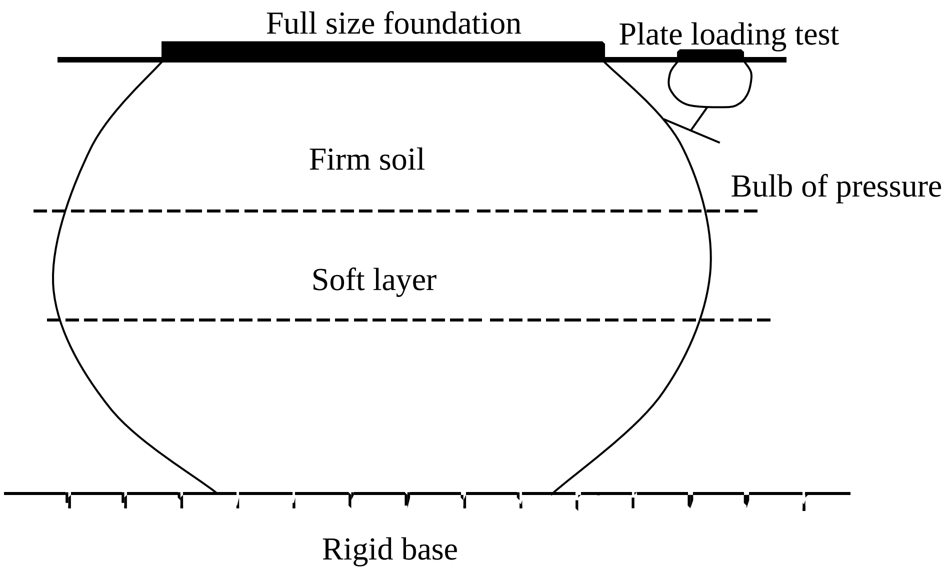


Figure 10.25 Illustration of how a plate loading test may give misleading results

A reasonable approximation of modulus of subgrade reaction k_s can be obtained from the allowable soil pressure q_{all} according to *Bowles* (1977). This way is presented on the assumption that the allowable soil pressure is based on some maximum amount of settlement s, including a factor of safety FS. Accordingly, the modulus of subgrade reaction k_s is given by

$$k_s = F\,S\,\frac{q_{all}}{s} \tag{10.119}$$

As an example, the modulus of subgrade reaction k_s [kN/m^3] for a settlement of s = 0.0254 [m] and a factor of safety FS = 3 can be taken as

$$k_s = 3 \frac{q_{all}}{0.0254} = 120\, q_{all} \tag{10.120}$$

In case of carrying out the analysis with constant modulus of subgrade reaction, it is recommended to determine the modulus of subgrade reaction from settlement calculation. More complicated analysis for irregular foundation on variable moduli of subgrade reactions is available in *ELPLA*. Furthermore, the moduli of subgrade reactions can be improved through the calculated contact pressures and settlements by iteration.

The following Table 10.4 shows the approximate average values of k_s according to *Wölfer* (1978). These values may be used only for primary calculation.

Table 10.4 Typical average values of moduli of subgrade reactions *ks* for selected soils

Type of soil	Modulus of subgrade reaction k_s [kN/m^3]
Peat	5000 - 10000
Fill of sand and gravel	10000 - 20000
Wet clayey soil	20000 - 30000
Moistured clay	40000 - 50000
Dry clay	60000 - 80000
Hard dry clay	100000
Coarse sand	80000 - 100000
Coarse sand + small amount of gravel	80000 - 100000
Fine gravel + small amount of gravel	80000 - 100000
Middle size gravel + fine sand	100000 - 120000
Middle size gravel + coarse sand	120000 - 150000
Large size gravel + coarse sand	150000 - 200000

10.14.6 Allowable bearing capacity of the soil q_{all}

The value of allowable bearing capacity of the soil is based on theoretical as well as experimental investigation. Such a value usually includes a factor of safety of 3 (q_{ult} = 3 q_{all}). This indicates that the design loads used in establishing the bearing capacity area of the foundation must be service loads with no reduction.

Approximate allowable bearing capacity q_{all} of common types of soils are listed in Table 10.5 according to *Bakhoum* (1986) and can be taken for primary calculations.

Table 10.5 Approximate allowable bearing capacity q_{ult} of common types of soils

Type of soil	Allowable bearing capacity q_{all} [kN/m^2]
Noncohesive soil	
Loose sand	100
Medium sand	200
Dense sand	500
Hard rock	5000
Cohesive soil	
Soft-medium clay	90
Stiff clay	150
Very stiff clay	300
Hard clay	500

10.14.7 Settlement reduction factor α

From experience the real consolidation settlements are different from those calculated. Settlements *s* are multiplied by a factor α according to German standard DIN 4019, page No. 1. According to this standard, the following reduction factors in Table 10.6 can be applied:

Table 10.6 Reduction factors α according to DIN 4019, page No. 1

Soil type	α
Sand and silt	0.66
Normally and slightly over consolidated clay	1.0
Heavily over consolidated clay	0.5 - 1

In *GEO Tools*, the moduli of compressibility E_s and W_s are divided by α as follows

$$\left.\begin{aligned} \overline{E}_s &= \frac{E_s}{\alpha} \\ \overline{W}_s &= \frac{W_s}{\alpha} \end{aligned}\right\} \tag{10.121}$$

In the final result, this process is equivalent to the following equation

$$\overline{S} = \alpha\, s \tag{10.122}$$

10.15 References

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