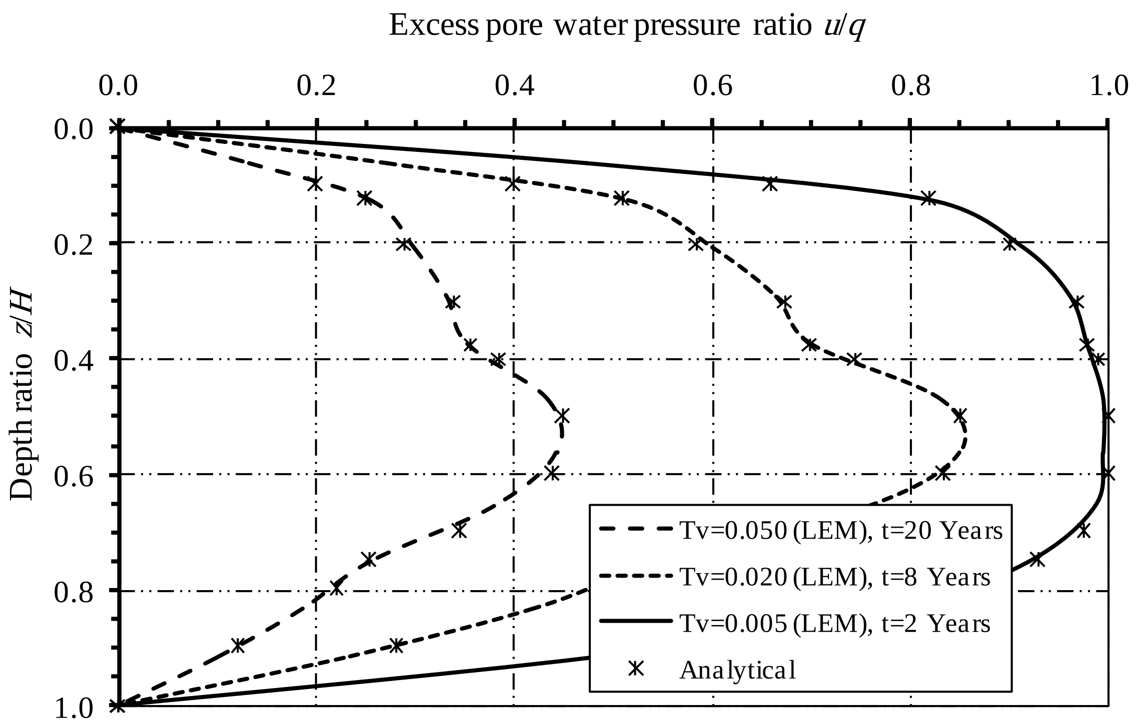
Degree of Consolidation
(Constant, Variable, and Cyclic Loadings)
by the
Program *GEO Tools*



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Preface

Various problems in geotechnical Engineering can be investigated by the program *GEO Tools*. The original version of *GEO Tools GEOTEC Office* was developed by Prof. M. Kany, Prof. M. El Gendy and Dr. A. El Gendy. After the death of Prof. Kany, Prof. M. El Gendy and Dr. A. El Gendy further developed the program to meet the needs of practice.

This book describes procedures and methods available in *GEO Tools* to analyze the problem of time-dependent consolidation of clay. The methods consider various analysis aspects of the consolidation problem such as, among others: uniform and time-dependent loading, cycling loading, nonlinear compressibility parameters, multi-layered soil, normal- and over-consolidated clay. The initial applied stress on the clay layers may be considered non-uniform.

GEO Tools has been developed for solving time-dependent settlement problems of clay layers using three different numerical methods:

- *L*ayer *E*quation *M*ethod (*LEM*), that was developed by *Herrmann*/ *El Gendy* (2014).
- *F*inite *D*ifference *M*ethod (*FDM*), that is the traditional solution of consolidation problems.
- *E*igen *V*alue *M*ethod *(EVM*).

Although a more practical and meshless method *LEM* was developed for the program, the old one of *FDM* was considered because many geotechnical engineers are familiar with it. The developed *LEM* method depends on selecting a number of nodes in the soil layers. Consequently, a better representation for applied stresses on the layers can be represented. *LEM* requires fewer equation terms, in which a few terms are sufficient to give good results.

El Gendy, O. (2016) had carried out a numerical modification on the semi-analytical solution of *Toufig and Ouria* (2009) to be applicable for multi-layered soil subjected to any variable stress along the depth of the soil using *LEM*. Some of verification examples for cyclic loading on multi-layered soil carried out by him are presented in this book.

Many tested examples are presented to verify and illustrate the available methods. Furthermore, an application for *LEM* on reloading time-dependent settlement of clay is presented in which a deep excavation is necessary for buildings with basements. In this case, the soil stress reduces due to excavation, and the reloading of the soil should be taken into account.

5 Degree of Consolidation

5.1 Finite Difference Method (*FDM*)

5.1.1 Introduction

The analytical solution is difficult to solve the consolidation problem, when the clay layer is subjected to an irregular distribution of initial excess pore water pressure. In this case, the use of a numerical method is fairly common. The oldest numerical method used for solving the consolidation problem is the Finite Difference Method, which was proposed by *Gibson and Lumb* (1953). The assumption of this method is that the one-dimensional consolidation equation and the boundary condition are approximated by finite difference formula. This numerical solution for one-dimensional consolidation is described in the next paragraphs.

5.1.2 Formulation of excess pore water pressure for a single layer

The basic differential equation for one dimensional consolidation of *Trezaghi's* consolidation theory is:

$$\frac{\partial u}{\partial t} = C_v \frac{\partial^2 u}{\partial z^2} \tag{5.1}$$

where:

u Pore water pressure at depth z, [kN/m^2]

t Time for which excess pore water pressure is computed, [sec]

C_v Coefficient of consolidation, [m^2/sec]

To solve this equation numerically, consider the clay layer shown in Figure 5.1 in which the soil layer is free drainage at its top and bottom. The layer of thickness H_o is divided into m equal intervals of thickness Δz.

According to Taylor's theorem:

$$u_i(t+\Delta t) = u_i(t) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3} + ... \tag{5.2}$$

Ignoring second derivatives and above, the time derivative can be approximated by:

$$\frac{\partial u}{\partial t} \approx \frac{u_i(t+\Delta t) - u_i(t)}{\Delta t} \tag{5.3}$$

where the index i indicates that the values refer to the excess pore water pressures in the point $z = z_i$.

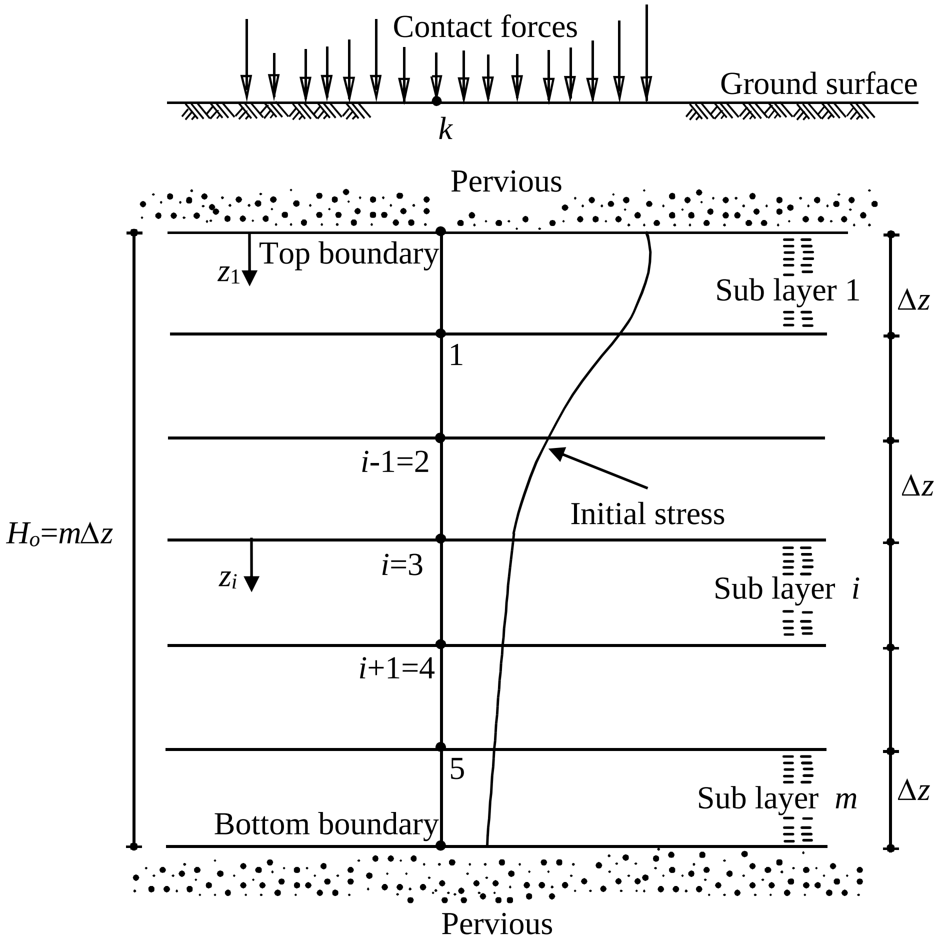


Figure 5.1 Excess pore water pressure under surface point *k* variations at time *t*

The second derivative with respect to *z* can be approximated by the same way:

$$\frac{\partial^2 u}{\partial z^2} \approx \frac{u_{i+1}(t) - 2u_i(t) + -u_{i-1}(t)}{\Delta z^2} \tag{5.4}$$

Substituting Eq. (5.3) and Eq. (5.4) in Eq. (5.1), gives:

$$u_i(t+\Delta t) = \alpha u_{i-1}(t) + (1-2\alpha)u_i(t) + \alpha u_{i+1}(t) \tag{5.5}$$

where $\alpha = \frac{C_v \Delta t}{\Delta z^2}$ is the operator of the equation, for convergence the value of the operator must not exceed 1/6.

This equation can be used for calculating the excess pore water pressure *u* in the grid points. The excess pore pressure *u* at time $t + \Delta t$ in the point *i* is calculated from that at time *t* in that point and in the two points just above (*i*+1) and just below (*i*-1). This mean that *u* at time *t* is required in order to calculate *u* at time $t + \Delta t$.

Equation (5.5) is written in matrix form for m grid points in z-direction. Then, the excess pore water pressure at any time for m nodes along the depth axis z can be expressed in a matrix form as:

$$\begin{Bmatrix} u_{top} \\ u_1 \\ u_2 \\ \cdots \\ u_m \\ u_{bot} \end{Bmatrix}_{t+\Delta t} = \begin{bmatrix} \alpha & 1-2\alpha & \alpha & 0 & 0 & 0 \\ 0 & \alpha & 1-2\alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1-2\alpha & \alpha & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \alpha & 1-2\alpha \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix} \begin{Bmatrix} u_{top} \\ u_1 \\ u_2 \\ \cdots \\ u_m \\ u_{bot} \end{Bmatrix}_{t} \tag{5.6}$$

where:

u_{top} Pore water pressure at the top of the layer, [kN/m^2]
u_{bot} Pore water pressure at the bottom of the layer, [kN/m^2]
u_i Pore water pressure at any depth i, [kN/m^2]
t Time for which excess pore water pressure is computed
H_o Layer thickness, [m]
Tc Consolidation time
Δt Time interval, $\Delta t = T_c/\omega$
Δz Depth interval, $\Delta z = H_o/m$
ω Number of time intervals
m Number of grid points
m+1 Number of depth intervals
α Operator, $\alpha = Cv\ \Delta t/\Delta z^2 \leq 1/6$

5.1.2.1 Initial Condition

The initial excess pore water pressure at time t=0 is required to solve the finite difference scheme. In one dimensional consolidation, the initial excess pore water pressure distribution u is equal to the distribution of the applied vertical stress σ on the clay layer, thus:

$$\{u_{top} \quad u_1 \quad u_2 \quad \ldots \quad u_m \quad u_{bottom}\}^T = \{\sigma_{top} \quad \sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_m \quad \sigma_{bottom}\}^T \tag{5.7}$$

5.1.2.2 Permeable Boundary

At a free draining boundary there is no barrier to the flow and so the pore pressure remains constant, thus the excess pore water pressure is zero. The soil layer is free drainage at its top and bottom. Therefore, the excess pore water pressure at the top and bottom of the layer drops to zero, $u_{top} = u_{bot} = 0$, at the first time interval. Equation (5.6) at t_1 for the shown clay layer in Figure 5.1 is rewritten as:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_1 = \begin{bmatrix} 1-2\alpha & \alpha & 0 & 0 & 0 \\ \alpha & 1-2\alpha & \alpha & 0 & 0 \\ 0 & \alpha & 1-2\alpha & \alpha & 0 \\ 0 & 0 & \alpha & 1-2\alpha & \alpha \\ 0 & 0 & 0 & \alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} \tag{5.8}$$

5.1.2.3 Impermeable Boundary

When the adjoining stratum at one boundary is impervious (Figure 5.2), there will be no flow across the impermeable boundary.

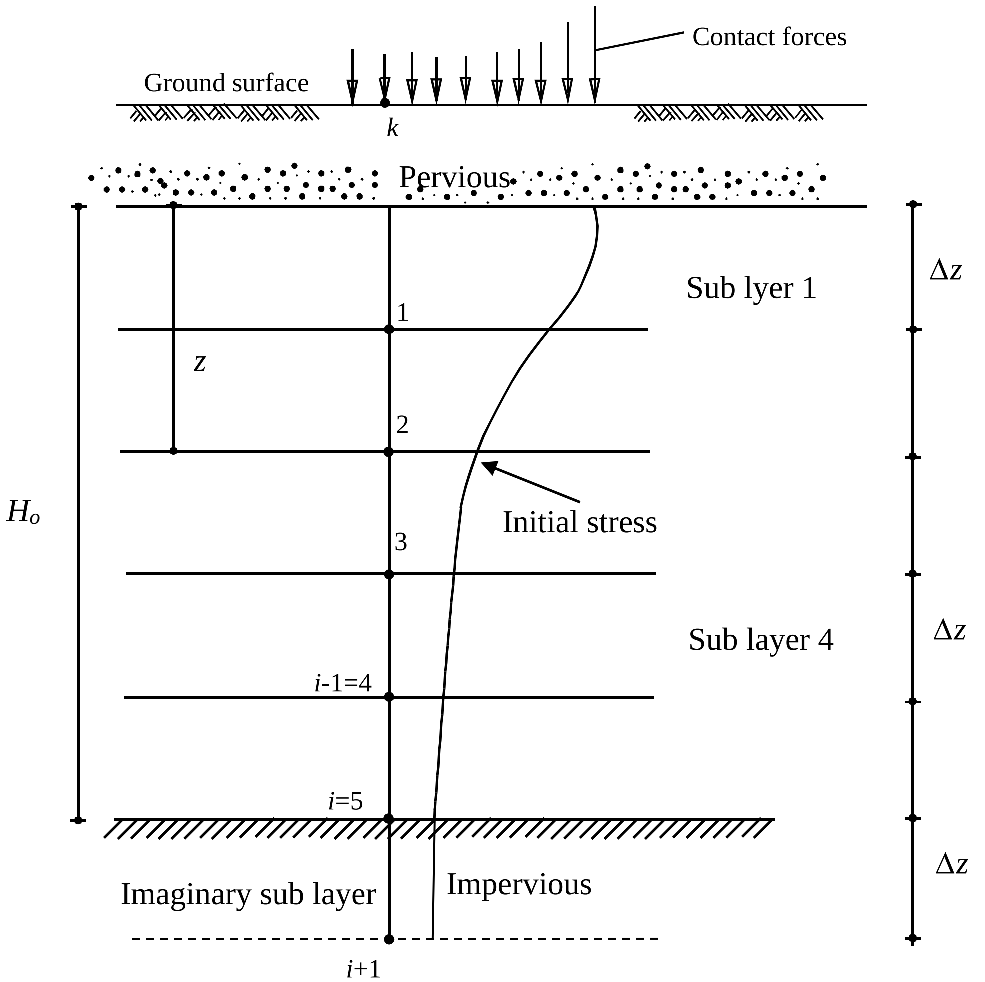


Figure 5.2 Clay layer with impermeable boundary at bottom

According to Darcy's Law at an impervious boundary:

$$\frac{\partial u}{\partial z} = 0 \tag{5.9}$$

That can be approximated by continuing the numerical subdivision by one more interval below $z=H_o$, so that in a point at a distance Δz below the lower boundary a value of the pore water pressure is defined, say $u_{i+1}(t)$. By requiring that $u_{i-1}(t) = u_{i+1}(t)$, the condition $\frac{\partial u}{\partial z} = 0$ is satisfied at the boundary.

Let $u_{i-1}(t) = u_{i+1}(t)$ in Eq. (5.5), therefore for a point i on an impermeable boundary the Eq. (5) can be modified as:

$$u_i(t+\Delta t) = 2\alpha u_{i-1}(t) + (1-2\alpha)u_i(t) \tag{5.10}$$

Equation (5.8) for a clay layer with impermeable boundary at bottom in this case becomes:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_1 = \begin{bmatrix} 1-2\alpha & \alpha & 0 & 0 & 0 \\ \alpha & 1-2\alpha & \alpha & 0 & 0 \\ 0 & \alpha & 1-2\alpha & \alpha & 0 \\ 0 & 0 & \alpha & 1-2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} \tag{5.11}$$

5.1.3 Formulation of excess pore water pressure for multi-layered system

Consider as an example the two-layered system of clay layers with single drainage shown in Figure 5.3. Each layer of thickness h_i has a different soil parameters. The vertical velocity of the flow in both layers must be the same at the interface between the two layers.

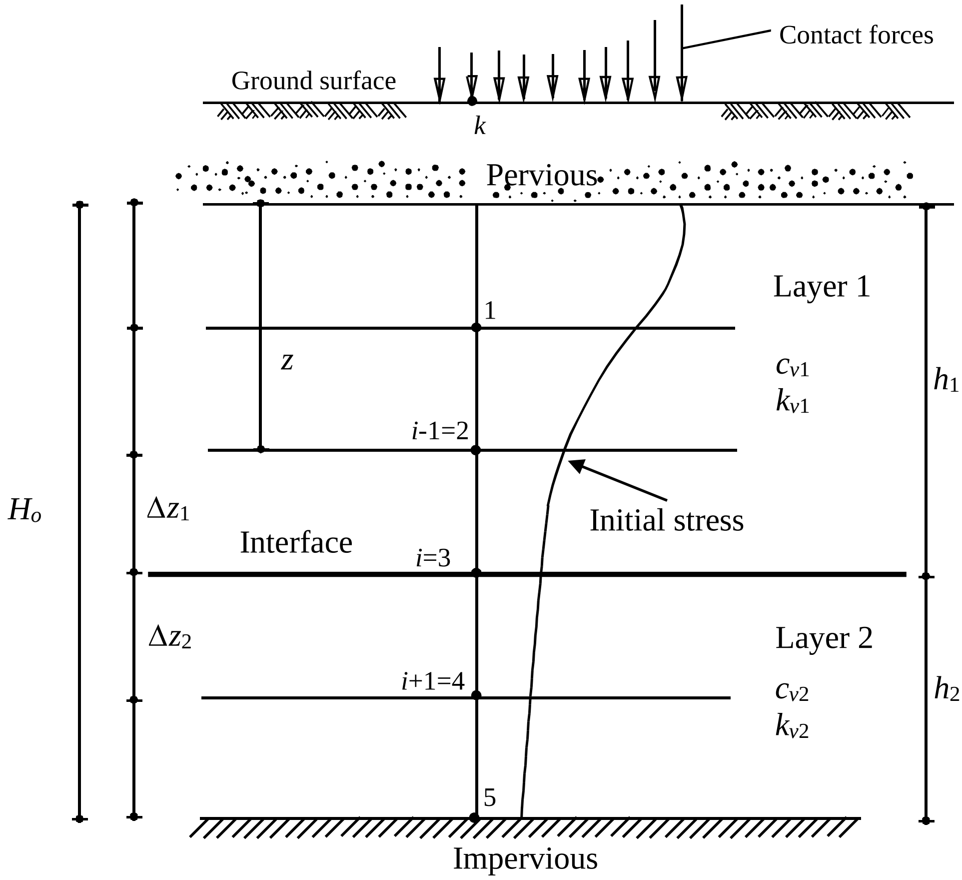


Figure 5.3 Two-layered system

Equating the velocity flow at layer interfaces:

$$k_{v1}\left(\frac{\partial u}{\partial z}\right)_1 = k_{v2}\left(\frac{\partial u}{\partial z}\right)_2 \tag{5.12}$$

At the interface i between layers 1 and 2, the velocity of the flow in the two layers may be expressed in difference as:

$$k_{v1}\left(\frac{-u_{i-1}+u_i}{\Delta z_1}\right) = k_{v2}\left(\frac{-u_i+u_{i+1}}{\Delta z_2}\right) \tag{5.13}$$

where:

u_{i-1} Pore water pressure in layer 1 before the interface i at depth h_1-Δz_1, [kN/m^2]
u_{i+1} Pore water pressure in layer 2 after the interface i at depth h_i+Δz_{i+1}, [kN/m^2]
u_i Pore water pressure at the interface, [kN/m^2]
Δz_j Depth interval in layers j, [m]
kv_j Coefficient of permeability of layer j, [m/sec]

Equation (5.13) may be rewritten as:

$$u_i = \frac{1}{(1+\beta_j)} u_{i-1} + \frac{\beta_j}{(1+\beta_j)} u_{i+1} \tag{5.14}$$

where

$$\beta_j = \frac{k_{j+1}\Delta z_j}{k_j \Delta z_{j+1}}$$
$$u_{i-1} = \alpha_1 u_{i-2}(t) + (1-2\alpha_1) u_{i-1}(t) + \alpha_1 u_i(t)$$
$$u_{i+1} = \alpha_2 u_i(t) + (1-2\alpha_2) u_{i+1}(t) + \alpha_2 u_{i+2}(t)$$

Thus,

$$\begin{aligned} u_i = {} & \frac{\alpha_1}{(1+\beta_j)} u_{i-2}(t) + \frac{(1-2\alpha_1)}{(1+\beta_j)} u_{i-1}(t) + \frac{\alpha_1 + \alpha_2\beta_j}{(1+\beta_j)} u_i(t) \\ & + \frac{(1-2\alpha_2)\beta_j}{(1+\beta_j)} u_{i+1}(t) + \frac{\alpha_2\beta_j}{(1+\beta_j)} u_{i+2}(t) \end{aligned} \tag{5.15}$$

For the two layers, let the operator α and time increment Δt are the same, therefore $\alpha_j = \alpha_{j+1} = C_{v\,j}\left(\frac{\Delta t}{\Delta z_j^2}\right) = C_{v\,j+1}\left(\frac{\Delta t}{\Delta z_{j+1}^2}\right)$, consequently $\beta_j = \frac{k_{j+1}}{k_j}\sqrt{\frac{C_{v\,j}}{C_{v\,j+1}}}$.

Equation (11) for two-layered system with impermeable boundary at bottom in this case becomes:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_1 = \begin{bmatrix} 1-2\alpha & \alpha & 0 & 0 & 0 \\ \alpha & 1-2\alpha & \alpha & 0 & 0 \\ \frac{\alpha}{(1+\beta_j)} & \frac{(1-2\alpha)}{(1+\beta_j)} & \frac{\alpha+\alpha\beta_j}{(1+\beta_j)} & \frac{(1-2\alpha)\beta_j}{(1+\beta_j)} & \frac{\alpha\beta_j}{(1+\beta_j)} \\ 0 & 0 & \alpha & 1-2\alpha & \alpha \\ 0 & 0 & 0 & 2\alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} \tag{5.16}$$

Now, any of the matrix equations (5.8), (5.11) or (5.16) of the excess pore water pressure can be rewritten in compacted matrix form as:

$$\{u\}_1 = [H]\{\sigma\} \tag{5.17}$$

where:

$\{u\}_1$ Excess pore water vector for the first time interval
$\{\sigma\}$ Vector of initial excess pore water
$[H]$ Operator matrix

The excess pore water pressure at any time can be computed explicitly as follows:

$$\begin{aligned}
&\text{at time } t = t_o + \Delta t && \{u\}_1 = [H]\{\sigma\} \\
&\text{at time } t = t_o + 2\Delta t && \{u\}_2 = [H]\{u\}_1 = [H][H]\{\sigma\} \\
&\text{at time } t = t_o + 3\Delta t && \{u\}_3 = [H]\{u\}_2 = [H][H][H]\{\sigma\}
\end{aligned} \tag{5.18}$$

and for the ω increment

$$\{u\}_\omega = [H]^\omega \{\sigma\} \tag{5.19}$$

5.2 Eigen Value Method (*EVM*)

5.2.1 Introduction

The traditional Finite Difference Method (*FDM*) is used for the solution of the consolidation problems, but the time stepping process in the solution is highly time consuming. An Eigen Value Method (*EVM*) is adapted for analyzing time-dependent settlement problems. This method is derived from the original finite deference solution of the consolidation problem out lined in the previous section. It obeys the same stability rules and time discretization of the finite difference solution of the problem. The numerical solution for one-dimensional consolidation by Eigen Value Method (*EVM*) for a homogenous clay layer may be found in the reference *Al-Kafaje* (1992). The discretion of the essential formulation of the consolidation problem by *EVM* is described in the next paragraphs.

5.2.2 Formulation of excess pore water pressure vector

5.2.2.1 Defining Eigenvalues and Eigenvectors

Consider $[A]$ is an $n \times n$ matrix. Then, the value λ is an eigenvalue of $[A]$ if there exists a non-zero vector $\{\varphi\}$ such that:

$$[A]\{\varphi\} = \lambda\{\varphi\} \tag{5.20}$$

In this case, vector $\{\varphi\}$ is called an eigenvector of $[A]$ corresponding to λ.

The corresponding eigenvectors $\{\varphi\}_r$ for eigenvalues λ_r can be found by solving a set of linear simultaneous equations as follows:

$$[A]\{\varphi\}_r = \lambda_r\{\varphi\}_r \tag{5.21}$$

where λ_r represent the eigenvalues of the basic equations $[A]$ $\{\varphi\}_r$, r=1, 2, 3...n.

5.2.2.2 Computing Eigenvalues and Eigenvectors

The equation $[A]\{\varphi\}=\lambda\{\varphi\}$can be rewritten as:

$$\big[[A]-\lambda[I]\big]\{\varphi\} = 0 \tag{5.22}$$

where $[I]$ is the $n \times n$ identity matrix.

In order for a non-zero vector $\{\varphi\}$ to satisfy this equation, $[A]-\lambda[I]=0$ must be not invertible. That is, the determinant of $[A]-\lambda[I]$ must be equal 0. Therefore, the eigenvalues of $[A]$ are the roots of the characteristic polynomial $p(\lambda)$:

$$p(\lambda) = \det.\big[[A]-\lambda[I]\big] \tag{5.23}$$

For each eigenvalue λ_r, the eigenvector $\{\varphi\}_r$ is obtained by solving the linear system $[A]-\lambda[I]=0$.

The above basic equation (19) may be rewritten for all n as:

$$\begin{aligned}
&[A]\{\varphi\}_1 = \lambda_1\{\varphi\}_1 \\
&[A]\{\varphi\}_2 = \lambda_2\{\varphi\}_2 \\
&[A]\{\varphi\}_3 = \lambda_3\{\varphi\}_3 \\
&..... \\
&\text{or} \\
&[A][\Phi] = [\Phi][\lambda]
\end{aligned} \tag{5.24}$$

Then, the matrix [A] is given:

$$[A] = [\Phi][\lambda][\Phi]^{-1} \tag{5.25}$$

where [Φ] is the square eigenvalue matrix:

$$[\Phi] = \left[\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \cdots \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_n \right]$$

and [λ] is the diagonal eigenvalue matrix

$$[\lambda] = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n \end{bmatrix}$$

The advantage of Eq. (24) is that raising the diagonal eigenvalue matrix [λ] to any power ω is carried out by raising its diagonal elements to that power for example.

$$[A]^{nt} = \left[\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \cdots \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_n \right] \begin{bmatrix} \lambda_1^{nt} & & & \\ & \lambda_2^{nt} & & \\ & & \cdots & \\ & & & \lambda_n^{nt} \end{bmatrix} \left[\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_1 \cdots \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_n \end{Bmatrix}_n \right]^{-1} \tag{5.26}$$

5.2.2.3 Computing excess pore water pressure by EVM

As, the new values of excess pore water pressure at any time can be computed explicitly as follows:

$$\{u\}_\omega = [H]^\omega \{\sigma\} \tag{5.27}$$

Then, applying the *EVM* on the modified operator matrix [H]$^\omega$, gives the explicit eigenvalue solution for the excess pore water pressure at time intervals ω.

$$\{u\}_{\omega} = [\Phi][\lambda]^{\omega}[\Phi]^{-1}\{\sigma\} \tag{5.28}$$

where:

$[H]=[\Phi][\lambda][\Phi]^{-1}$	Operator matrix
$[\Phi]$	Square eigenvalue matrix
$[\lambda]$	Diagonal eigenvalue matrix

It is convenient to express the time in terms of the dimensionless parameter such as time factor. The time intervals ω can be expressed as:

$$\omega = \frac{m^2 C_v T_c}{\alpha H_d^2} = \frac{m^2 T_v}{\alpha} \tag{5.29}$$

where:

m No. of studied points

T_v Time factor, $T_v = \dfrac{c_v T_c}{H_d^2}$

H_d Length of drainage, [m].

For double drainage $H_d = \dfrac{H_o}{2}$ while for single drainage $H_d = H_o$

and $\alpha = \dfrac{C_v \Delta t}{\Delta z^2} = \dfrac{C_v\left(\dfrac{T_c}{\omega}\right)}{\left(\dfrac{H_d}{m}\right)^2} = \dfrac{m^2 C_v T_c}{\omega H_d^2}$

Expressing the time intervals by ω, the new values of excess pore water pressure at any time can be computed explicitly as follows:

$$\{u\}_{\omega} = [H]^{\omega}\{\sigma\} \tag{5.30}$$

Applying the Eigenvalue Method *EVM* on the operator matrix $[H]^{\omega}$, gives the explicit Eigenvalue solution for the excess pore water pressure at time intervals ω.

$$\{u\}_{\omega} = [\Phi][\lambda]^{\omega}[\Phi]^{-1}\{\sigma\} \tag{5.31}$$

As it is mentioned before, the advantage of Eq. (5.31) is that raising the diagonal eigenvalue matrix [λ] to any power ω is carried out by raising its diagonal elements to that power. Therefore, by the conventional Finite Difference Method *FDM* the large number of calculation suffers from round off error at each time interval. Consequently, the analysis progress of *EVM* allows to determine the excess pore water pressure at any time without needing to compute intermediate values. It is also possible to calculate the excess pore water pressure at a friction of a time interval.

5.2.2.4 Eigenvalues and Eigenvectors for a clay layer with 5 grid nodes

As an example for a clay layer with pervious boundaries at top and bottom of m =5 grid nodes with an operator α = 1/6, the eigenvalues may be given directly by:

$$\lambda_r = 1 - 4\alpha \sin^2\left(\frac{r\pi}{2m}\right), \qquad r = 1..., m \tag{5.32}$$

where r is the interior node for which the Eigenvalue is needed, r = 1 to m.

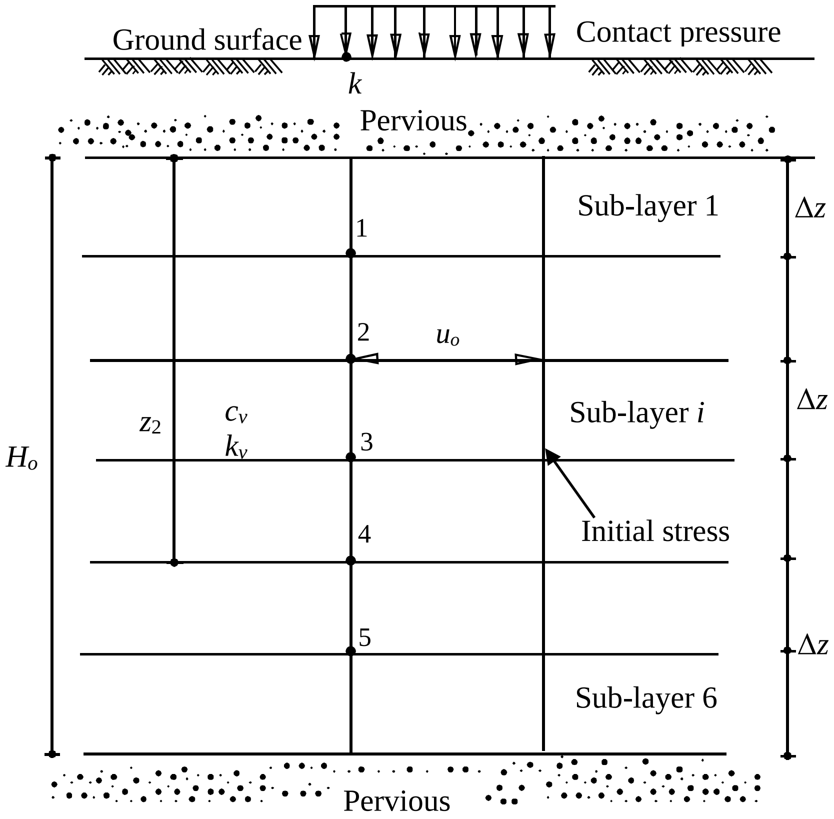


Figure 5.4 Single layer with 6 sub-layers

Then, the eigenvalues for points 1 to 5 are:

$$\begin{aligned}
\lambda_1 &= 1-\frac{4}{6}\sin^2\left(\frac{1\times\pi}{2\times 6}\right) = 0.9553\\
\lambda_2 &= 1-\frac{4}{6}\sin^2\left(\frac{2\times\pi}{2\times 6}\right) = \frac{5}{6}\\
\lambda_3 &= 1-\frac{4}{6}\sin^2\left(\frac{3\times\pi}{2\times 6}\right) = \frac{2}{3}\\
\lambda_4 &= 1-\frac{4}{6}\sin^2\left(\frac{4\times\pi}{2\times 6}\right) = \frac{1}{2}\\
\lambda_5 &= 1-\frac{4}{6}\sin^2\left(\frac{2\times\pi}{2\times 6}\right) = 0.3780
\end{aligned} \tag{5.33}$$

The corresponding square eigenvalue matrix [Φ] and its inverse are:

$$[\Phi] = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ \sqrt{3} & -1 & 0 & 1 & -\sqrt{3} \\ 2 & 0 & -1 & 0 & 2 \\ \sqrt{3} & 1 & 0 & -1 & -\sqrt{3} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.34}$$

$$[\Phi]^{-1} = \frac{1}{12}\begin{bmatrix} 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\ -3 & -3 & 0 & 3 & 3 \\ 4 & 0 & -4 & 0 & 4 \\ -3 & 3 & 0 & -3 & 3 \\ 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 \end{bmatrix} \tag{5.35}$$

Considering the symmetry of the problem due to the double drainage at the top and bottom of the layer and a uniform initial excess pore water pressure u_o on the layer, Eq. (31), yields to:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{\omega} = \frac{u_o}{6}\begin{bmatrix} \left(2+\sqrt{3}\right) & 2 & \left(2-\sqrt{3}\right) \\ \left(3+2\sqrt{3}\right) & 0 & \left(3-2\sqrt{3}\right) \\ \left(4+2\sqrt{3}\right) & -2 & \left(4-2\sqrt{3}\right) \end{bmatrix}\begin{Bmatrix} \lambda_1^{\omega} \\ \lambda_3^{\omega} \\ \lambda_5^{\omega} \end{Bmatrix} \tag{5.36}$$

5.3 Layer Equation Method (*LEM*)

5.3.1 Introduction

Most of the available meshless methods for time-dependent settlement problems depend on deriving an algebraic equation for each layer; some of these methods were introduced by *Lee et al.* (1992), *Xie et al.* (2002, 2004 and 2005), *Zhuang et al.* (2005) and *Morris* (2002). In the solution, the derived equation has an infinite number of series functions with an infinite number of coefficients. Also the original solution of the 1-D consolidation problem presented by *Terzaghi* (1925) was a formula with infinite series. Infinite series may be lead to oscillation. Furthermore, the methods assumed a uniformly initial applied stress on the clay layers or regular shapes of stress such as a triangular shape (*Singh* 2005). In a better case, the stress was assumed as a continuous function in depth (*Lee et al.* 1992). The reason is that it is difficult to generate infinite coefficients to represent the applied variable stress on the clay.

In this book, the Layer Equation Method (*LEM*) developed by *Herrmann* / *El Gendy* (2014) for analyzing time-dependent settlement problems is considered. *LEM* depends on selecting a number of nodes in the clay layers along the *z*-axis. Consequently, a better representation for applied stress on soil layers can be represented. The method is also ideal for using a stress coefficient technique, which may be extended to study the interaction of irregular loaded areas on the surface or contact pressure due to foundation rigidity. *LEM* requires fewer equation terms, in which fewer terms are sufficient to give excellent results compared to the available closed-form solution of time-dependent settlement problems. However, algebraic equations of clay layers are developed from an initial stress applied to a specified number of grid nodes, which can represent the excess pore water pressure at any node on the layers.

LEM is used to investigate the behavior of the excess pore water pressure when the clay changes from an over-consolidated state to a normally-consolidated state during the consolidation process because the stress applied to the clay layers varies with time. These states were studied for a single clay layer by *Xie et al.* (2008). They had determined the moving depth of the interface between over- and normally-consolidated zones in a layer. The layer is considered to have an impervious base. The initial load applied to the layer in each interval increment of time was uniform. Also, the initial vertical effective stress due to weight of the entire layer itself was uniform. Besides, only two coefficients of consolidation were considered; one for the normally-consolidated zone; and the other for the over-consolidated zone. They had considered this case as a double-layered soil. In fact, the initial applied stress generated on the soil from the surface loading is not uniform throughout the clay depth. It is greater near the surface than at the base. In addition, maybe the clay has a pervious top and base. It is also known that the initial vertical effective stress increases with depth. This means that at any sub-layer within the clay, the state may change from over- to normally-consolidated, especially for a thick clay layer. The analysis in this book takes into account the nonlinear response of the excess pore water pressure due to the change of compressibility and permeability of the soil during the consolidation process.

5.3.2 Constant Loading

5.3.2.1 Formulation of Excess Pore Water Pressure

5.3.2.1.1 Defining Basic Functions

To formulate the analysis, the loaded area on the surface is divided into triangular elements as shown in Figure 5.5. Then the contact pressure is represented by a series of contact forces Q_j on the element nodes. The soil under the loaded area may consist of multi-layered system of clay with different soil parameters and is divided into *n* sub-layers with *r* nodes as shown in Figure 5.6. Stress coefficients for the nodes under the loaded area due to contact forces can be determined as described by *El Gendy* (2006).

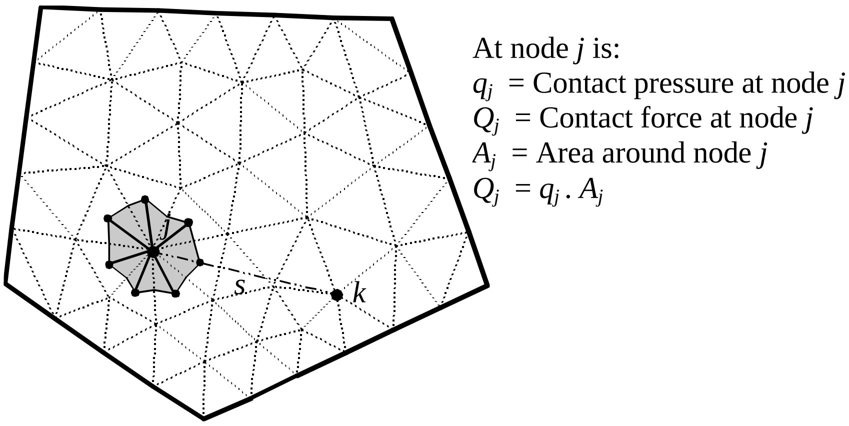


Figure 5.5 Loaded area with mesh of elements on the surface

According to *El Gendy* (2006), for a set of grid nodes of *m* contact forces Q_j at the surface, the vertical stress σ_l in a node depth *l* under the surface node *k* is attributed to stresses caused by all of the contact forces on that node:

$$\sigma_l = \sum_{j=1}^{m} f_{l,j}\, Q_j \qquad (5.37)$$

where $f_{l,j}$ is the stress coefficient of node *l* due to the contact force at node *j* on the surface, [1/m^2]. It depends only on the geometry of the loaded area and the soil layer.

Each layer in Figure 5.6 has different soil parameters and geometries. k_{vi} [m/Year], Cv_i [Year/m^2], m_{vi} [m^2/kN], z_i [m] and h_i [m] are the coefficient of permeability, the coefficient of consolidation, the coefficient of volume change, the depth and the thickness of the *i*th soil layer, respectively. *H* [m] is the total thickness of the clay layers.

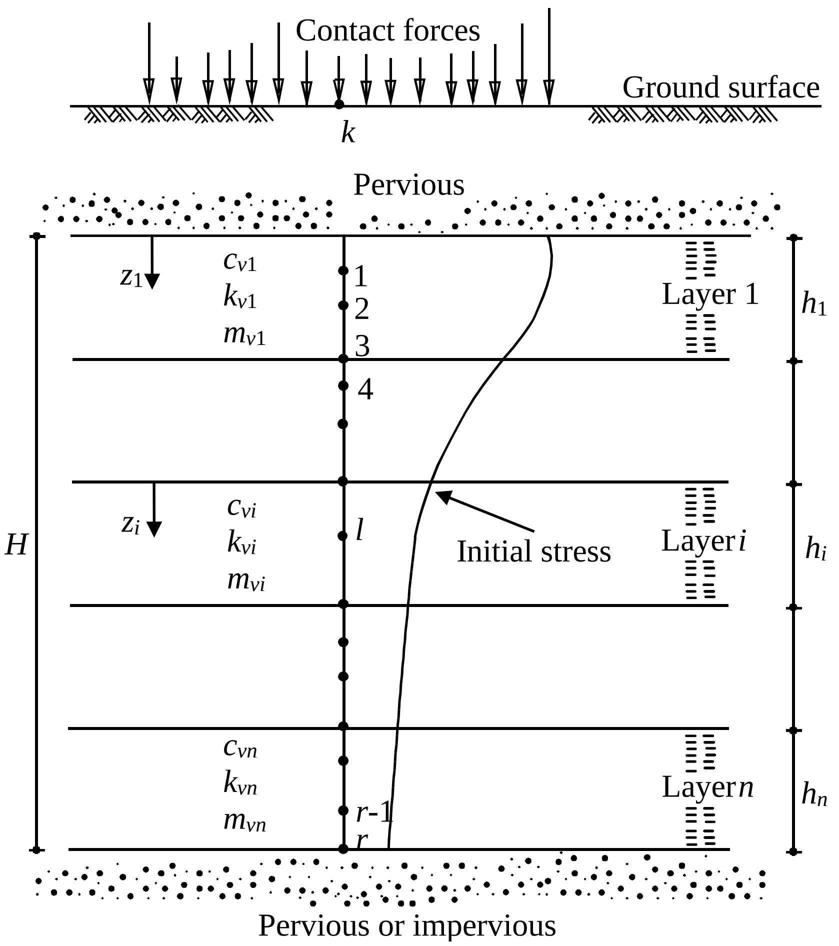


Figure 5.6 Multi-layered system

At a time t = 0, Eq. (5.37), for the entire clay layers at a section in the z-axis passing through point k, in matrix form becomes:

$$\{\sigma\}_o = [f]\{Q\}_o \tag{5.38}$$

where:

$\{\sigma\}_o$ Initial vertical stress vector at time t = 0

$[f]$ Stress coefficient matrix

$\{Q\}_o$ Initial contact force vector at time t = 0.

The solution depends on choosing a formula that represents the excess pore water pressure along the z-axis and satisfies the boundary conditions. A partial differential equation such as the consolidation equation can be solved and expressed in a series of N terms as:

$$u(z,t) = \sum_{j=1}^{N} C_j \varphi_j(z) \psi_j(t) \tag{5.39}$$

where:

$u(z, t)$	Excess pore water pressure at any vertical depth z and time t, [kN/m^2]
$\varphi_j(z)$	Set of basic functions in the variable z only
$\psi_j(t)$	Coefficients of basic functions in the variable t only
C_j	Constants of basic functions
N	Number of function terms (Number of studied nodes)
z	Vertical coordinate, [m]
t	Time for which excess pore water pressure is computed, [year].

Coefficients and constants of basic functions can be obtained by selecting a set of N arbitrary nodes and their function values $u(z, t)$. The basic functions are chosen to satisfy the boundary condition. The boundary conditions for double drainage are $u(0, t) = 0$ and $u(H_o, t) = 0$, while those for single drainage are $u(0, t) = 0$ and $\partial u(H_o, t)/\partial z = 0$. To select N arbitrary nodes, each layer is divided into m_i sub-layers with depth increment $\Delta z_i = h_i/m_i$, which gives a total r nodes. For a pervious bottom boundary the excess pore water pressure at the bottom boundary is known and equal to zero. Therefore, studied nodes in this case are less than those of an impervious bottom boundary by a node. The number of studied nodes will be $N = r$-1 for a pervious bottom boundary, while that for an impervious bottom boundary will be $N = r$. Suitable basic functions for excess pore water pressure problems are as follows:

$$\varphi_j(z) = A_{ij} \sin(\mu_i \lambda_j \xi_i) + B_{ij} \cos(\mu_i \lambda_j \xi_i) \tag{5.40}$$

and corresponding coefficients are:

$$\psi_j(t) = \exp(-\mu_i^2 \lambda_j^2 T_{vi}) \tag{5.41}$$

where:

ξ_i	Local depth ratio for layer i, $\xi_i = z_i/h_i$
A_{ij} and B_{ij}	Coefficients of basic functions
λ_j	Differential equation operator
μ_i	Parameter of the coefficient of consolidation and thickness, $\mu_i = (h_i / h_1)\sqrt{C_{v1}/C_{vi}}$
T_{vi}	Time factor, $T_{vi} = c_{vi} t / h_i^2$.

Now, the equation for excess pore water pressure $u_i(z, t)$ for layer i in a multi-layered system may be expressed as follows:

$$u_i(z,t) = \sum_{j=1}^{N} C_j \left[A_{ij} \sin(\mu_i \lambda_j \xi_i) + B_{ij} \cos(\mu_i \lambda_j \xi_i)\right] \exp(-\mu_i^2 \lambda_j^2 T_{vi}) \tag{5.42}$$

To satisfy the condition of the governing differential equation for 1-D consolidation (*Terzaghi's* equation) $C_{vi} \dfrac{\partial^2 u_i}{\partial z^2} = \dfrac{\partial u_i}{\partial t}$, the following equations should be satisfied:

$$\mu_1^2\lambda_j^2 T_{v1} = \mu_2^2\lambda_j^2 T_{v2} = ... = \mu_n^2\lambda_j^2 T_{vn} = \omega_j t \qquad (5.43)$$

where $\omega_j = \frac{c_{v1}}{h_1^2}\lambda_j^2$

Eq. (5.42) may be written for N studied nodes in a matrix form as:

$$\{u\} = [\Phi]\,[E_v]^t\{C\} \qquad (5.44)$$

where:
$\{u\}$ Vector of the excess pore water pressure u_j, j=1 to N
$\{C\}$ Vector of constants C_j, j=1 to N
$[\Phi]$ Matrix of basic functions
$[E_v]$ Diagonal square matrix of the exponential functions.

where matrix $[\Phi]$ for double drainage boundaries is given by:

$$[\Phi] = \begin{bmatrix} B_{21} & B_{22} & B_{23} & ... & ... & ... & B_{2n} \\ B_{31} & B_{32} & B_{33} & ... & ... & .... & B_{3n} \\ B_{41} & B_{42} & B_{43} & ... & ... & ... & B_{4n} \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ B_{(n-1)1} & B_{(n-1)2} & B_{(n-1)3} & ... & ... & ... & B_{(n-1)n} \\ |B_{n1}| & |B_{n2}| & |B_{n3}| & ... & ... & ... & |B_{nn}| \end{bmatrix} \qquad (5.45)$$

and matrix $[\Phi]$ for single drainage boundary is given by:

$$[\Phi] = \begin{bmatrix} B_{21} & B_{22} & B_{23} & ... & ... & ... & B_{2n} \\ B_{31} & B_{32} & B_{33} & ... & ... & .... & B_{3n} \\ B_{41} & B_{42} & B_{43} & ... & ... & ... & B_{4n} \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ B_{n1} & B_{n2} & B_{n3} & ... & ... & ... & B_{nn} \\ B_{(n+1)1} & B_{(n+1)2} & B_{(n+1)3} & ... & ... & ... & B_{(n+1)n} \end{bmatrix} \qquad (5.46)$$

while the diagonal square matrix $[E_v]$ is given by:

$$[E_v] = \text{diag}\left[\exp(-\omega_1) \quad \exp(-\omega_2) \quad \exp(-\omega_3) \quad ... \quad \exp(-\omega_N)\right] \qquad (5.47)$$

5.3.2.1.2 Determining Coefficients A_{ij} and B_{ij}

Relations among coefficients A_{ij} and B_{ij} can be obtained using interface and boundary conditions. Equating the excess pore water pressures $u_i(z_i, t) = u_{i+1}(z_i, t)$ at layer interfaces, leads to:

$$A_{ij}\sin(\mu_i\lambda_j\xi_i) + B_{ij}\cos(\mu_i\lambda_j\xi_i) = A_{(i+1)j}\sin(\mu_{i+1}\lambda_j\xi_{i+1}) + B_{(i+1)j}\cos(\mu_{i+1}\lambda_j\xi_{i+1}) \qquad (5.48)$$

while equating the vertical velocity of flow $k_{vi}\left(\frac{\partial u}{\partial z}\right)_i = k_{v(i+1)}\left(\frac{\partial u}{\partial z}\right)_{i+1}$ at layer interfaces, leads to:

$$\frac{\mu_i k_{vi} h_{i+1}}{\mu_{(i+1)} k_{v(i+1)} h_i}\left[A_{ij}\cos(\mu_i\lambda_j\xi_i) - B_{ij}\sin(\mu_i\lambda_j\xi_i)\right] = A_{(i+1)j}\cos(\mu_{i+1}\lambda_j\xi_{i+1}) - B_{(i+1)j}\sin(\mu_{i+1}\lambda_j\xi_{i+1}) \qquad (5.49)$$

At the interface $\xi_i = 1$ and $\xi_{i+1} = 0$, Eqs. (5.48) and (5.49) then become:

$$A_{ij}\sin(\mu_i\lambda_j) + B_{ij}\cos(\mu_i\lambda_j) = B_{(i+1)j} \qquad (5.50)$$

$$\frac{\mu_i k_{vi} h_{i+1}}{\mu_{(i+1)} k_{v(i+1)} h_i}\left[A_{ij}\cos(\mu_i\lambda_j) - B_{ij}\sin(\mu_i\lambda_j)\right] = A_{(i+1)j} \qquad (5.51)$$

Satisfying free drainage at the top $u_1(0, t) = 0$, requires that:

$$A_{1j} = 1 \text{ and } B_{1j} = 0 \qquad (5.52)$$

From Eqs. (5.51) and (5.52), coefficients A_{ij} and B_{ij} can be expressed as:

$$\{R\}_{i+1} = [\theta]_i \{R\}_i \qquad (5.53)$$

where

$$\begin{aligned} \{R\}_i &= \{A_{ij} \quad B_{ij}\}^T,\ i = 1, 2, ..., n \\ [\theta]_i &= \begin{bmatrix} \eta_i\cos(\mu_i\lambda_j) & -\eta_i\sin(\mu_i\lambda_j) \\ \sin(\mu_i\lambda_j) & \cos(\mu_i\lambda_j) \end{bmatrix} \end{aligned} \qquad (5.54)$$

and:

$$\eta_i = \frac{\mu_i k_{vi} h_{i+1}}{\mu_{(i+1)} k_{v(i+1)} h_i} = \frac{m_{vi}}{m_{v(i+1)}}\sqrt{\frac{C_{vi}}{C_{v(i+1)}}} \qquad (5.55)$$

The vector $\{R\}_n$ is obtained from boundary conditions of the two cases of single and double drainages. Applying boundary conditions at the base where u=0 for double drainage and $\partial u/\partial z$=0 for single drainage, thus gives:

$$[S_d]\{R\}_n = 0 \tag{5.56}$$

where the matrix $[S_d]$ is given by:

$$\begin{aligned}
[S_d] &= [\sin(\mu_n \lambda_j) \quad \cos(\mu_n \lambda_j)] \quad \text{for double drainage} \\
[S_d] &= [\cos(\mu_n \lambda_j) \quad -\sin(\mu_n \lambda_j)] \quad \text{for single drainage}
\end{aligned} \tag{5.57}$$

From Eqs. (5.54) and (5.57), the following characteristic equation in the unknown Eigen values λ_j (differential equation operators) can be obtained:

$$[S_d][\theta]_{n-1}\{R\}_{n-1} = 0 \tag{5.58}$$

The operator λ_j is the positive root of the above characteristic equation. Substituting the value of λ_j obtained from Eq. (5.58) into Eq. (5.53), gives coefficients A_{ij} and B_{ij}.

5.3.2.1.3 Determining Constants C_j

Constants C_j can be found using the initial condition $u_j(z, 0) = u_o(z)$. Consider a system of linear equations at a set of N grid nodes at time $t = 0$ as follows:

$$\{u\}_o = [\Phi]\{C\} \tag{5.59}$$

where $\{u\}_o$ is the vector of initial excess pore water pressure

Substituting Eq. (5.59) into Eq. (5.44), gives the following matrix equation for excess pore water pressure:

$$\{u\} = [\Phi]\,[E_v]^t[\Phi]^{-1}\{u\}_o \tag{5.60}$$

The advantage of Eq. (5.60) is that raising the diagonal matrix $[E_v]$ to any power of time t is carried out by raising its diagonal elements E_{vj} to that power. This equation is similar to the equation of *EVM* but it has different eigenvalues because the deriving of both are different..

5.3.2.2 Degree of Consolidation

Integrating Eq. (5.42) over the entire layer i, gives the average excess pore water pressure Δu_i at any time factor in that layer as follows:

$$\Delta u_i = \frac{1}{h_i}\int_0^{h_i} u_i(z,t)\,dz = \frac{1}{\mu_i}\sum_{j=1}^{N}\frac{C_j}{\lambda_j}\{A_{ij}[1-\cos(\mu_i\lambda_j)] + B_{ij}\sin(\mu_i\lambda_j)\}\,\exp(-\omega_j t) \tag{5.61}$$

The initial average stress $\Delta\sigma_{oi}$ in a layer i is given by:

$$\Delta\sigma_{oi} = \frac{1}{h_i}\int_0^{h_i}\sigma_{oi}(z)\,dz \tag{5.62}$$

where $\sigma_{oi}(z)$ is the initial stress in a layer i due to a foundation load. [kN/m^2].

The degree of consolidation U_p and U_s at the required time t can be obtained from either the stress:

$$U_p = 1 - \frac{\sum_{i=1}^{n} \Delta u_i h_i}{\sum_{i=1}^{n} \Delta \sigma_{oi} h_i} \tag{5.63}$$

or the settlement:

$$U_s = 1 - \frac{\sum_{i=1}^{n} m_{vi} \Delta u_i h_i}{\sum_{i=1}^{n} m_{vi} \Delta \sigma_{oi} h_i} \tag{5.64}$$

5.3.3 Variable (Linear) Loading

In practice, the total load on clay under a structure is applied over a period of time. In this case, the total load of construction on the surface q_c can be applied gradually over a time t_c as shown in Figure 5.7. The governing equation for 1-D consolidation, taking into account the variable loading with construction time as indicated by *Lee et al.* (1992), can be expressed as:

$$c_{vi} \frac{\partial^2 u_i}{\partial z^2} = \frac{\partial u_i}{\partial t} - \frac{d\sigma_i}{dt} \tag{5.65}$$

An analytical solution for Eq. (5.65) is difficult. To determine the excess pore water pressure, the integral can be evaluated by a series of M steps-load increment at the surface Δq at interval of times Δt (Figure 5.7). The load increment at the surface Δq will lead to an increment of vertical stress $\delta\sigma_i$ at node i.

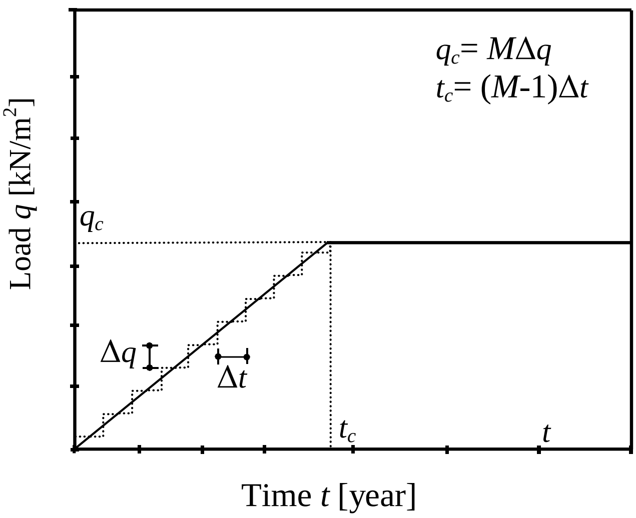


Figure 5.7 Applied load in a series of steps

For determining the excess pore water pressure due to step load increments, the excess pore water pressure induced by the previous load is obtained, and at the same time the excess pore water pressure induced by the additional load is determined. The results of this process may be expressed explicitly as:

$$\left.\begin{array}{l}\{u\}_1 = [\Phi][E_v]^{\Delta t}[\Phi]^{-1}\{\delta u\}_o \\ \{u\}_2 = [\Phi][E_v]^{\Delta t}[\Phi]^{-1}\{\{u\}_1 + \{\delta u\}_o\} \\ \{u\}_3 = [\Phi][E_v]^{\Delta t}[\Phi]^{-1}\{\{u\}_2 + \{\delta u\}_o\} \\ \ldots\ldots \\ \{u\}_{t_c} = [\Phi][E_v]^{\Delta t}[\Phi]^{-1}\{\{u\}_{(M-2)} + \{\delta u\}_o\} \\ \{u\}_t = [\Phi][E_v]^{t_r}[\Phi]^{-1}\{\{u\}_{t_c} + \{\delta u\}_o\}\end{array}\right\} \tag{5.66}$$

In Eq. (5.66), the total load is applied by M steps of equal load increment. Therefore, the additional initial pore water pressure vectors in all steps are the same and equal to $\{\delta u\}_o = \frac{1}{M}\{u\}_o$.

Now the vector of pore water pressure at time t may be written as:

$$\begin{aligned}\{u\}_t &= [\Phi][E_v]^{t_r+(M-1)\Delta t}[\Phi]^{-1}\{\delta u\}_o + [\Phi][E_v]^{t_r+(M-2)\Delta t}[\Phi]^{-1}\{\delta u\}_o \\ &+ \ldots. + \\ &+ [\Phi][E_v]^{t_r+2\Delta t}[\Phi]^{-1}\{\delta u\}_o + [\Phi][E_v]^{t_r+\Delta t}[\Phi]^{-1}\{\delta u\}_o + [\Phi][E_v]^{t_r}[\Phi]^{-1}\{\delta u\}_o\end{aligned} \tag{5.67}$$

where:

Δt Time interval, [Year], $\Delta t = t_c/(M-1)$

t_c Construction time, [year]; $t_r = t - t_c$, [year]

$\{\delta u\}_o$ Vector of additional initial pore water pressure

$\{u\}_{tc}$ Vector of pore water pressure at time t_c

$\{u\}_k$ Vector of pore water pressure at interval k.

Replacing t_r by t-t_c in the above equation gives:

$$\begin{aligned}\{u\}_t &= [\Phi][E_v]^t[\Phi]^{-1}\{\delta u\}_o + [\Phi][E_v]^{t-\Delta t}[\Phi]^{-1}\{\delta u\}_o \\ &+ [\Phi][E_v]^{t-2\Delta t}[\Phi]^{-1}\{\delta u\}_o + \ldots + [\Phi][E_v]^{t-(M-1)\Delta t}[\Phi]^{-1}\{\delta u\}_o\end{aligned} \quad (5.68)$$

Replacing $\{\delta u\}_o$ by $\frac{1}{M}\{u\}_o$ in the above equation and rewriting the equation gives:

$$\{u\}_t = \frac{1}{M}[\Phi][E_v]^t\begin{bmatrix}[I]+[E_v]^{-\Delta t}+[E_v]^{-2\Delta t}+[E_v]^{-3\Delta t} \\ +\ldots+[E_v]^{-(M-1)\Delta t}\end{bmatrix}[\Phi]^{-1}\{u\}_o \quad (69)$$

Eq. (5.69) is rewritten in matrix form as:

$$\{u\}_t = [\Phi]\,[E_v]^t\,[D]\,[\Phi]^{-1}\{u\}_o \quad (5.70)$$

where $[D]$ is a diagonal square matrix. The diagonal elements of the matrix $[D]$ are defined by:

$$D_j = \frac{1}{M}\left[1+\exp(\omega_j\Delta t)+\exp(2\omega_j\Delta t)+..+\exp((M\text{ -}1)\omega_j\Delta t)\right] \quad (5.71)$$

The summation of the above series when $M=\infty$ can be estimated as follows:

From the principle of mathematics, the summation of the following geometric series can be given by:

$$1+r+r^2+r^3+r^4+\ldots+r^{n-1} = \sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \quad (5.72)$$

Thus the summation of the series in Eq. (5.71) is given by:

$$D_j = \frac{1}{M}\frac{1-\exp(M\omega_j\Delta t)}{1-\exp(\omega_j\Delta t)} \quad (5.73)$$

Substituting the value of $M = \frac{t_c}{\Delta t}+1$ in Eq. (5.73), gives:

$$D_j = \frac{1}{\dfrac{t_c}{\Delta t}+1}\,\frac{1-\exp\left(\left\{\dfrac{t_c}{\Delta t}+1\right\}\omega_j \Delta t\right)}{1-\exp(\omega_j \Delta t)} \tag{5.74}$$

or

$$D_j = \frac{1-\exp(\omega_j t_c + \omega_j \Delta t)}{\dfrac{t_c}{\Delta t}-\dfrac{t_c}{\Delta t}\exp(\omega_j \Delta t)+1-\exp(\omega_j \Delta t)} \tag{5.75}$$

or

$$D_j = \frac{1-\exp(\omega_j t_c)\times\exp(\omega_j \Delta t)}{\dfrac{t_c}{\Delta t}-\dfrac{t_c}{\Delta t}\exp(\omega_j \Delta t)+1-\exp(\omega_j \Delta t)} \tag{5.76}$$

The exponential function $\exp(\omega_j \Delta t)$ can be defined by the following power series:

$$\exp(\omega_j \Delta t) = \sum_{n=0}^{\infty}\frac{(\omega_j \Delta t)^n}{n!} = 1+\omega_j \Delta t + \frac{(\omega_j \Delta t)^2}{2!}+\frac{(\omega_j \Delta t)^3}{3!}+.... \tag{5.77}$$

As the value of Δt is very small, then terms of Δt having power equal and more than 2 can be neglected, then:

$$\exp(\omega_j \Delta t) \approx 1+\omega_j \Delta t \tag{5.78}$$

Substituting the value of $\exp(\omega_j \Delta t)$ in Eq. (5.76), gives:

$$D_j = \frac{1-\exp(\omega_j t_c)\times(1+\omega_j \Delta t)}{\dfrac{t_c}{\Delta t}-\dfrac{t_c}{\Delta t}(1+\omega_j \Delta t)+1-(1+\omega_j \Delta t)} \tag{5.79}$$

or

$$D_j = \frac{1-\exp(\omega_j t_c)-\exp(\omega_j t_c)\times\omega_j \Delta t}{\dfrac{t_c}{\Delta t}-\dfrac{t_c}{\Delta t}-\dfrac{t_c}{\Delta t}\omega_j \Delta t+1-1-\omega_j \Delta t} \tag{5.80}$$

or

$$D_j = \frac{1-\exp\left(\omega_j t_c\right)-\exp\left(\omega_j t_c\right)\times \omega_j \Delta t}{-\omega_j t_c - \omega_j \Delta t} \tag{5.81}$$

The value of Δt tends to zero when $M = \infty$. Therefore, the summation of the sires in Eq. (5.71) is given by:

$$D_j = \frac{\exp\left(\omega_j t_c\right)\text{-}1}{\omega_j t_c} \tag{5.82}$$

Now, the equation for excess pore water pressure $u_i(z, t)$ for layer i in a multi-layered system becomes:

$$u_i(z,t) = \sum_{j=1}^{N} D_j C_j \left[A_{ij} \sin\left(\mu_i \lambda_j \xi_i\right) + B_{ij} \cos\left(\mu_i \lambda_j \xi_i\right)\right] \exp(\omega_j t) \tag{5.83}$$

and the average excess power water pressure Δu_i becomes:

$$\Delta u_i = \frac{1}{\mu_i} \sum_{j=1}^{N} \frac{D_j C_j}{\lambda_j} \left\{A_{ij} \left[1-\cos\left(\mu_i \lambda_j\right)\right] + B_{ij} \sin\left(\mu_i \lambda_j\right)\right\} \exp\left(-\omega_j t\right) \tag{5.84}$$

5.3.4 Cyclic loading

5.3.4.1 Introduction

Cyclic loadings are often applied to clay layers under structures subject to loading and unloading circumstances such as silos and tanks. *Toufig and Ouria* (2009) presented a semi-analytical method to determine the pore water pressure and degree of consolidation for a rectangular cyclic loading, considering the effect of the change of the consolidation coefficient of the soil layer. In the method, changes in the consolidation coefficient are applied by modifying the loading and unloading durations using a Virtual Time. Based on the superimposing rule a set of continuous static loads in specified times are used instead of the cyclic load in the transformed time space. Each full cycle of loading is replaced by a pair of static loads with different signs. Based on the *Terzaghi's* theory the pore-water pressure distribution and the degree of consolidation are calculated for each static load and the results are superimposed. *Toufig and Ouria* (2009) verified the solution by carrying out a set of laboratory consolidation tests under cyclic load.

El Gendy, O. (2016) had carried out a numerical modification on the semi-analytical solution of *Toufig and Ouria* (2009) to be applicable for multi-layered soil subjected to any variable stress along the depth of the soil using *LEM*. To illustrate the possibility of *LEM* to handle cyclic loading, three types of cyclic loadings are considered as shown in Figure 5.8. The change of compressibility of soil under cyclic loading can be described as shown in Figure 5.9. This numerical solution for cyclic loading on multi-layered soil is described in the next paragraphs.

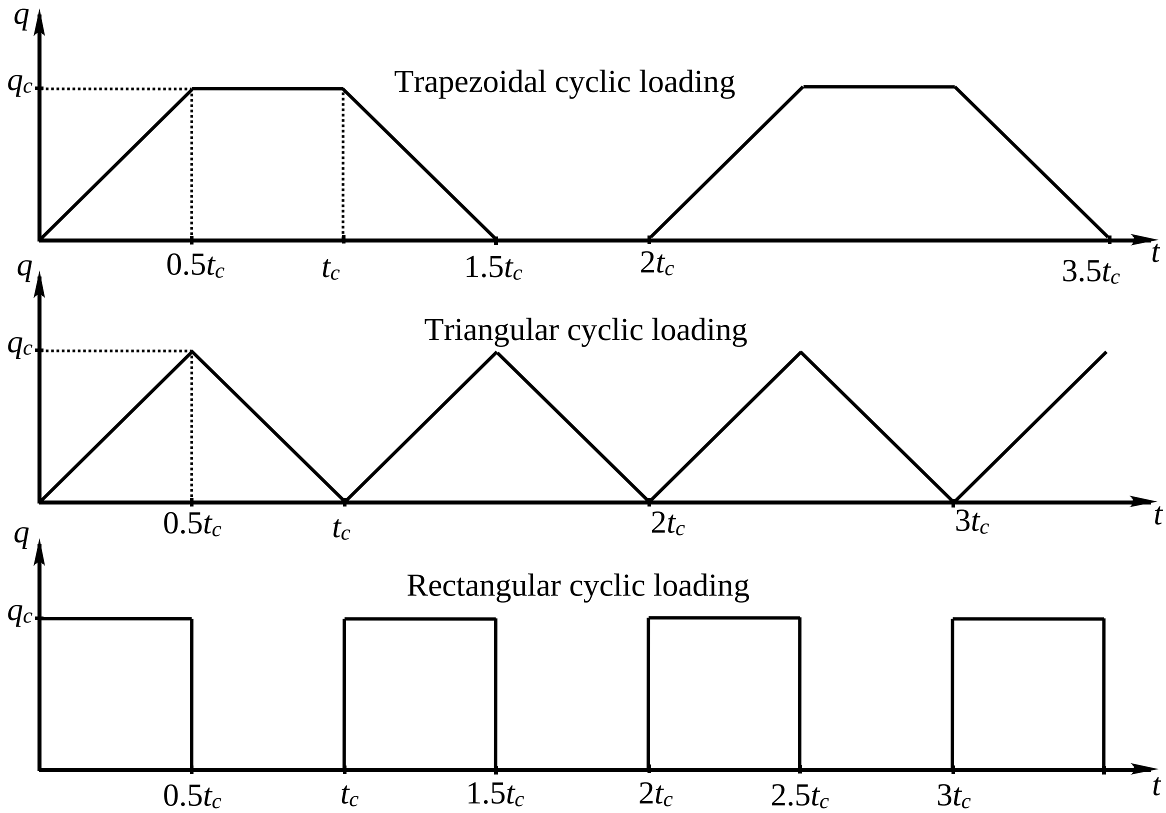


Figure 5.8 Cyclic loading types

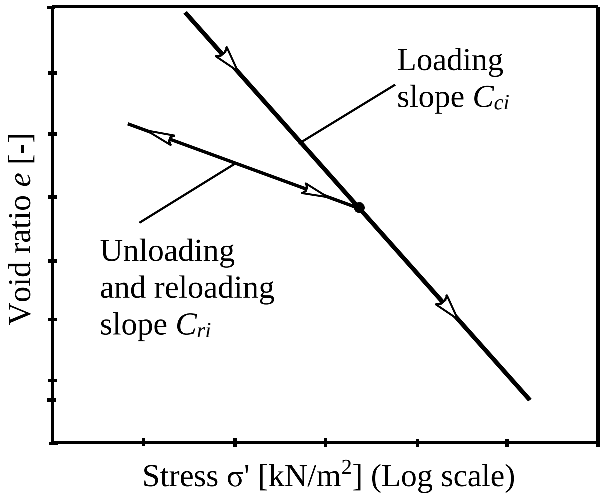


Figure 5.9 Relationship between void ratio and stress (loading, reloading and unloading cases)

5.3.5 Virtual time method for modeling rectangular cyclic loading

To illustrate the solution of the consolidation of the clay under cyclic loading using the virtual time method in a simple way, consider the bilinear model shown in Figure 5.10, which describes one cycle of rectangular cyclic loading as an example. In the model, the coefficients of volume change m_v and permeability k_v of the clay changes during the loading and unloading half cycles. The coefficient of consolidation Cv is a function of these parameters and changes in each cycle of loading. The coefficient of consolidation is assumed to have only two different values in the state of normally consolidated NC or overconsolidated OC as indicated before , where $C_{v(\mathrm{NC})} = \beta\ C_{v(\mathrm{OC})}$.

In Figure 5.10, at first half cycle, clay is in NC state and stress path is according to [1-2] route. During the unloading process of all half cycles, clay is at OC state and stress path is according to [2-3] route. After the first full cycle, in the next loading half cycles, stress path will be according to [3-4-5] route. Position of point 4 is the same as the preconsolidation pressure σ_c application point, which represents the maximum degree of consolidation that the clay obtained in the previous cycle. This σ_c increases by increasing number of cycles and reaches to a point where the clay stays in OC state during the entire loading phases, which is called steady-state. The clay is in the NC state when the degree of consolidation is greater than its previous values (according to routes [1-2] and [4-5]). It is OC when it does not have the maximum of the previous values (according to the route [2-3-4]).

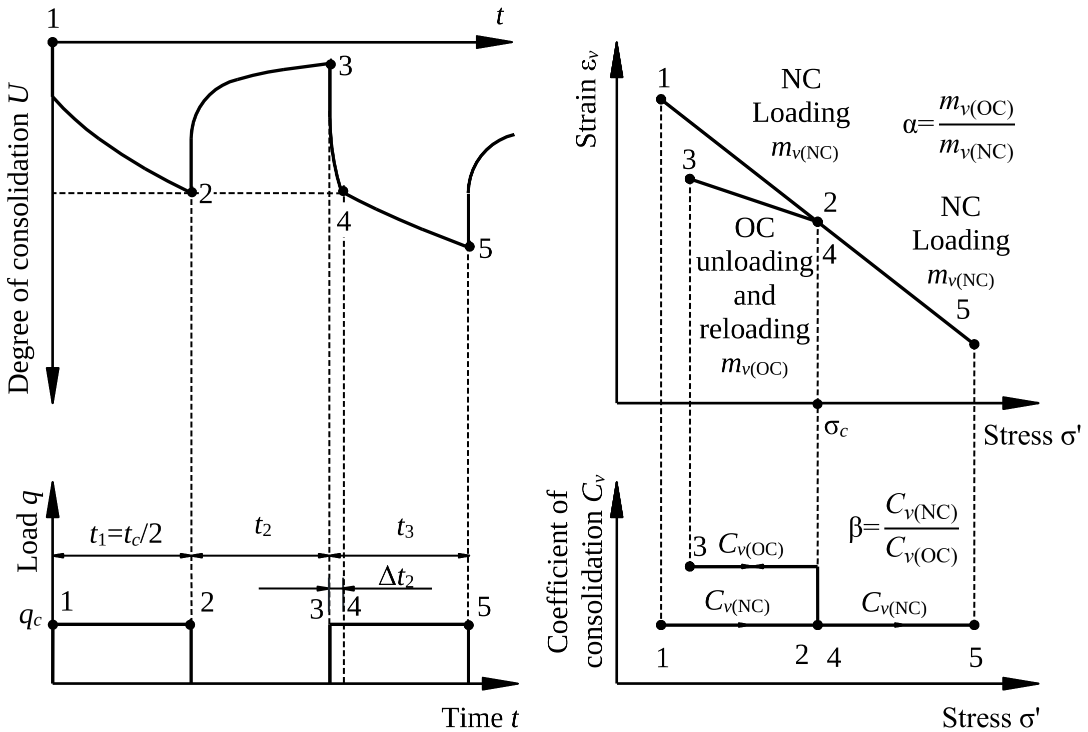


Figure 5.10 Plastic behavior of soil under cyclic loading

5.3.5.1.1 Definition of virtual time

Cyclic loading calculation requires to use two different values of coefficient of consolidation, the value of C_v in routes [1-2] and [4-5] is equal to $C_{v(NC)}$ and it is equal to $C_{v(OC)}$ in route [2-3-4]. Since the time factor T_v, is a linear function of coefficient of consolidation C_v and time t, it means that the equal variation of both factors would cause same changes in the results. In order to obtain the results considering the varying C_v, any changes of the C_v is applied to t and C_v is assumed to be constant as described in the following time factor equation:

$$T_v = \frac{(kC_v)t}{H_d^2} = \frac{C_v(kt)}{H_d^2} = \frac{C_v t'}{H_d^2}, \qquad t' = kt \tag{5.85}$$

where:

H_d Length of the drainage pass
t Real time
t' Virtual time and k can be any factor.

This idea introduces a transformation function, where a clay layer in which C_v is variable, constant C_v can be substituted in an adjusted time space. During the time period of the unloading half cycles (route [2-3 in Figure 5.10] where the clay is in OC state, the value of C_v is different from its value in NC state. In this case, the calculation can be carried out during unloading periods by $C_{v(NC)}$ and a virtual time t' using the Eq. (115). Therefore, the equivalent time for unloading half cycles may be defined as:

$$t_N^{'} = \frac{t_c}{2\beta}, \qquad N = 2, 4, 6 \tag{5.86}$$

where t_c is the period time of the cycle and β is the virtual time factor, which are introduced in Figure 5.10 and N is the number of the half cycle.

5.3.5.1.2 Determining the time portion of each phase

After the first full cycle (a loading and an unloading half cycle) as indicated in Figure 5.11, the clay is in OC state until the degree of consolidation $U_{\Delta 2}$ reaches the previous maximum degree of consolidation, which is equal to the degree of consolidation at the end of the last loading phase U_{c1}. The time portion of each loading phase Δt_N shown in Figure 5.11 to get points similar to point 4 in every half cycle of loading can be replaced by:

$$\Delta t_N^{'} = \frac{\Delta t_N}{\beta} \tag{5.87}$$

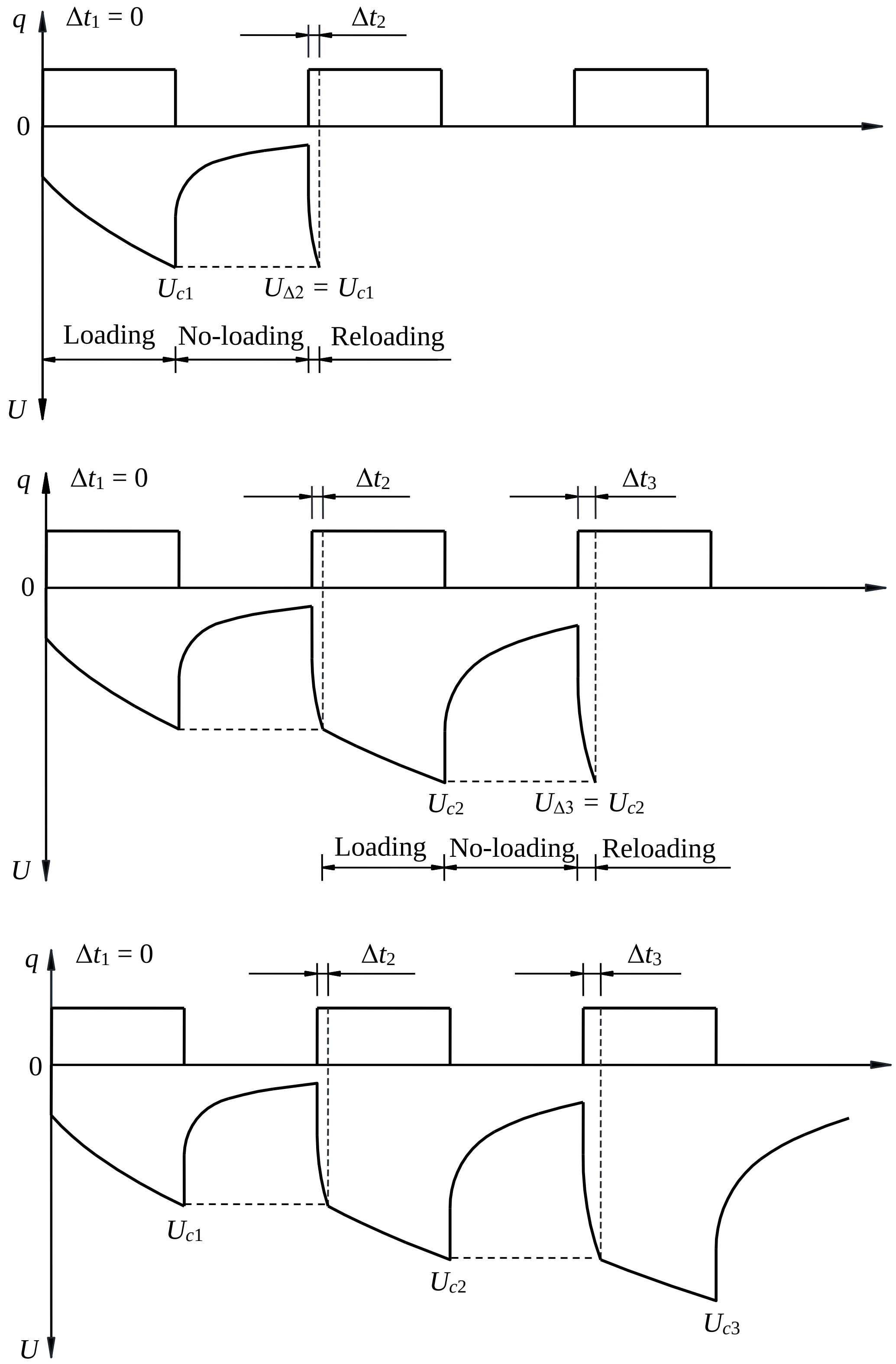


Figure 5.11 Determining the value of Δt_N

5.3.5.2 Rectangular Cyclic Loading under Loading and Reloading

Figure 5.12 shows rectangular cyclic loading adapted by the superimposing rule. In this type of loading, Δt_N is the time portion of each loading half cycle in which the soil is in OC state (according to route [3-4]) and then becomes NC. In order to define the virtual time transformation function for loading half cycles, the value of Δt_N must be known. On the other hand, superimposing rule can be used to replace a cyclic loading by a set of static loads. As shown in Figure 5.12, the cyclic loading system in the real time space is adapted in the virtual time space, each full cycle of cyclic load is replaced by a pair of static loads with plus and minus signs. The vector of pore water pressure for rectangular cyclic loading at a period of time t_c can be determined as follows:

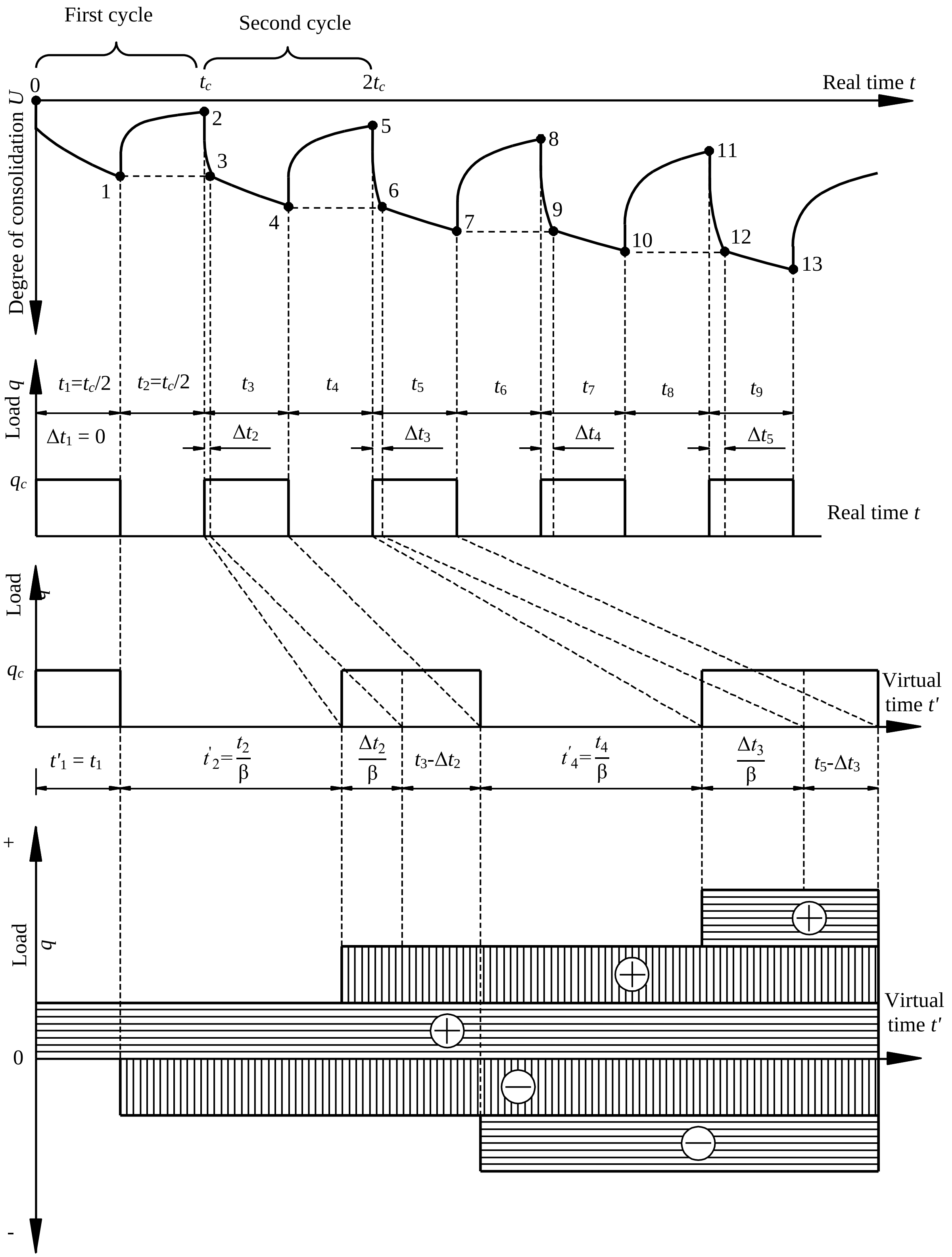


Figure 5.12 Rectangular cyclic loading adapted by the superimposing rule (Toufig & Ouria (2009))

At the first halve cycle (point 1), the pore water pressure vector is:

$$\{u\}_1 = [\Phi]\ [E_v]^{\frac{t_c}{2}}\ [\Phi]^{-1}\{u\}_o \tag{5.88}$$

At the second halve cyclic (point 2), the pore water pressure vector is:

$$\{u\}_2 = [\Phi]\ [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)}\ [\Phi]^{-1}\{u\}_o - [\Phi]\ [E_v]^{\frac{t_c}{2\beta}}\ [\Phi]^{-1}\{u\}_o \tag{5.89}$$

At the end of interval time Δt_2 (point 3), the pore water pressure vector is:

$$\{u\}_3 = [\Phi]\ [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)+\frac{\Delta t_2}{\beta}}\ [\Phi]^{-1}\{u\}_o - [\Phi]\ [E_v]^{\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}}\ [\Phi]^{-1}\{u\}_o + [\Phi]\ [E_v]^{\frac{\Delta t_2}{\beta}}\ [\Phi]^{-1}\{u\}_o \tag{5.90}$$

At the middle of the second cycle (point 4), the pore water pressure vector is:

$$\begin{aligned}\{u\}_4 &= [\Phi]\ [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)}\ [\Phi]^{-1}\{u\}_o - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)}[\Phi]^{-1}\{u\}_o \\ &+ [\Phi]\ [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)}\ [\Phi]^{-1}\{u\}_o\end{aligned} \tag{5.91}$$

At the end of the second cycle (point 5), the pore water pressure vector is:

$$\begin{aligned}\{u\}_5 &= [\Phi]\ [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [\Phi]\ [E_v]^{\left(\frac{t_c}{2}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} \\ &+ [\Phi]\ [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}}\ [\Phi]^{-1}\{u\}_o - [\Phi]\ [E_v]^{\frac{t_c}{2\beta}}\ [\Phi]^{-1}\{u\}_o\end{aligned} \tag{5.92}$$

Equations (5.89) to (5.92) may be rewritten as:

$$\{u\}_2 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)} - [E_v]^{\frac{t_c}{2\beta}}\right][\Phi]^{-1}\{u\}_o \tag{5.93}$$

$$\{u\}_3 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)+\frac{\Delta t_2}{\beta}} - [E_v]^{\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}} + [E_v]^{\frac{\Delta t_2}{\beta}}\right][\Phi]^{-1}\{u\}_o \tag{5.94}$$

$$\{u\}_4 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} + [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)}\right][\Phi]^{-1}\{u\}_o \tag{5.95}$$

$$\{u\}_5 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} + [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\frac{t_c}{2\beta}}\right][\Phi]^{-1}\{u\}_o \tag{5.96}$$

In general, the pore water pressure vector at the end of Δt_{nc} of the n_c cycle is given by (such as point

3):

$$\{u\}_{3n_c-3}=[\Phi]\left[\sum_{i=1}^{2n_c-1}(-1)^{(i+1)}[E_v]^{\left(T_i-\frac{t_c}{2\beta}-\frac{t_c}{2}+\Delta t_{n_c}\right)}\right][\Phi]^{-1}\{u\}_o \tag{5.97}$$

while, the pore water pressure vector at the middle of the n_c cycle is given by (such as point 4):

$$\{u\}_{3n_c-2}=[\Phi]\left[\sum_{i=1}^{2n_c-1}(-1)^{(i+1)}[E_v]^{\left(T_i-\frac{t_c}{2\beta}\right)}\right][\Phi]^{-1}\{u\}_o \tag{5.98}$$

and the pore water pressure vector at the end of the n_c cycle is given by (such as point 5):

$$\{u\}_{3n_c-1}=[\Phi]\left[\sum_{i=1}^{2n_c}(-1)^{(i+1)}[E_v]^{T_i}\right][\Phi]^{-1}\{u\}_o \tag{5.99}$$

where:

$$T_i=\left(2n_c(1+\beta)+2-\sum_{k=1}^{i}\left(1-(-1)^k\right)-\beta\sum_{k=1}^{i}\left(1-(-1)^{(k+1)}\right)\right)\frac{t_c}{4\beta}$$
$$+\frac{(1-\beta)}{2\beta}\left(2\sum_{k=1}^{n_c}\Delta t_k-\sum_{k=1}^{i}\left(1-(-1)^{(k+1)}\right)\Delta t_{\left(\frac{1-(-1)^{(k+1)}}{4}\right)k}\right)$$

5.3.5.3 Trapezoidal Cyclic Loading

The vector of pore water pressure for trapezoidal cyclic loading at a period of time $2t_c$ can be determined as follows (Figure 5.13):

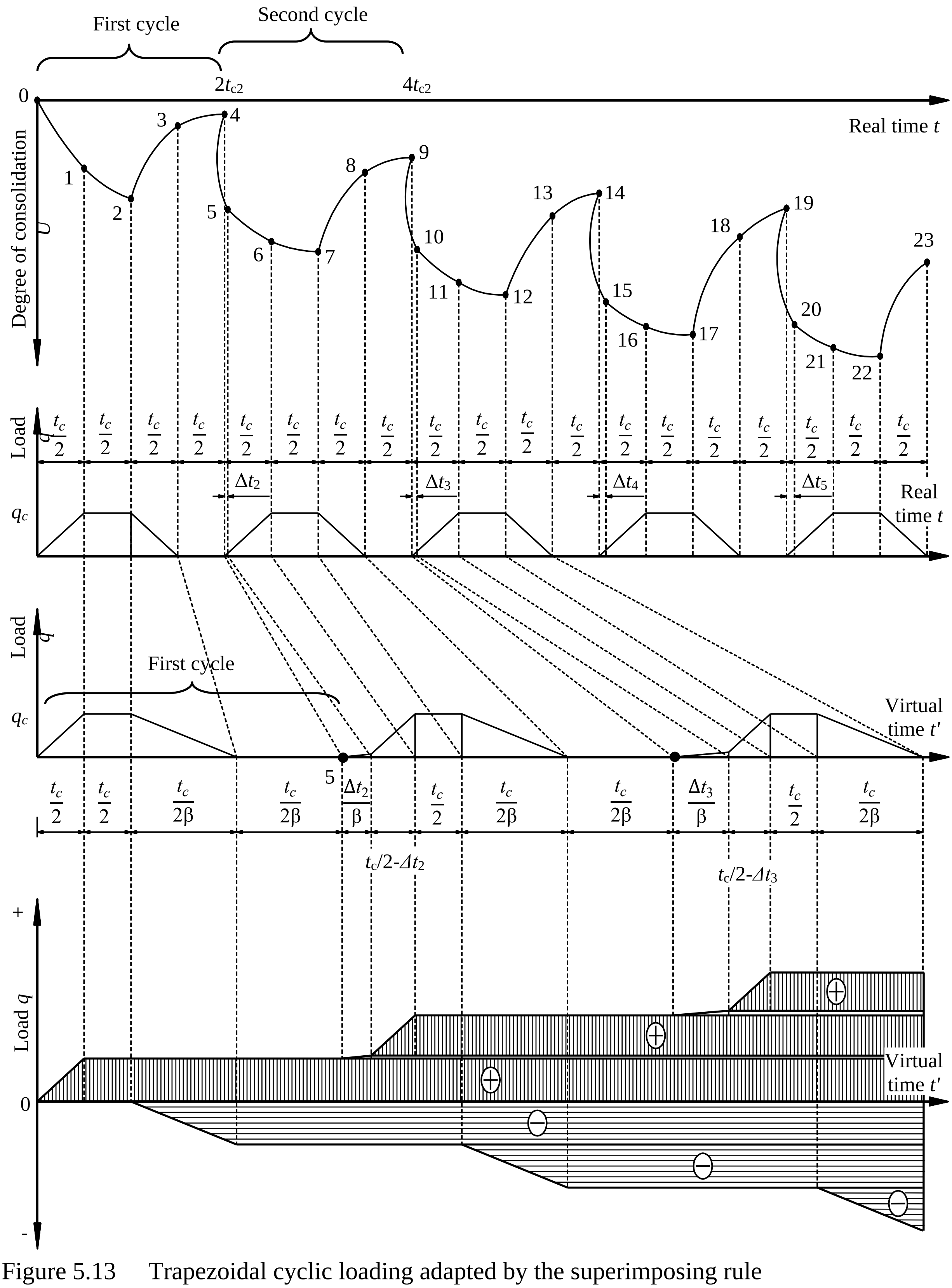


Figure 5.13 Trapezoidal cyclic loading adapted by the superimposing rule

At the first halve cycle (point 1), the pore water pressure vector is:

$$\{u\}_1 = [\Phi]\,[E_v]^{\frac{t_c}{2}}\,[D]\,[\Phi]^{-1}\{u\}_o \tag{5.100}$$

At the second halve cyclic (point 2), the pore water pressure vector is:

$$\{u\}_2 = [\Phi]\,[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)}[D][\Phi]^{-1}\{u\}_o - [\Phi]\,[E_v]^{\frac{t_c}{2\beta}}[D][\Phi]^{-1}\{u\}_o \tag{5.101}$$

At the end of interval time Δt_2 (point 3), the pore water pressure vector is:

$$\begin{aligned}\{u\}_3 &= [\Phi]\,[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)+\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{u\}_o - [\Phi]\,[E_v]^{\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{u\}_o\\ &+[\Phi]\,[E_v]^{\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{\Delta q_2\}\end{aligned} \tag{5.102}$$

or

$$\begin{aligned}\{u\}_3 &= [\Phi]\,[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)+\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{u\}_o - [\Phi]\,[E_v]^{\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{u\}_o\\ &+\frac{\Delta t_2}{t_c}[\Phi]\,[E_v]^{\frac{\Delta t_2}{\beta}}[D][\Phi]^{-1}\{u\}_o\end{aligned} \tag{5.103}$$

At the middle of the second cycle (point 4), the pore water pressure vector is:

$$\begin{aligned}\{u\}_4 &= [\Phi]\,[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)}[D][\Phi]^{-1}\{u\}_o - [\Phi]\,[E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)}[D][\Phi]^{-1}\{u\}_o\\ &+\frac{\Delta t_2}{t_c}[\Phi]\,[E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)}[D]^2\,[\Phi]^{-1}\{u\}_o+\left(1-\frac{\Delta t_2}{t_c}\right)[\Phi]\,[E_v]^{\left(\frac{t_c}{2}-\Delta t_2\right)}[D][\Phi]^{-1}\{u\}_o\end{aligned} \tag{5.104}$$

At the end of the second cycle (point 5), the pore water pressure vector is:

$$\begin{aligned}\{u\}_5 &= [\Phi]\,[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}}[D][\Phi]^{-1}\{u\}_o - [\Phi]\,[E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}}[D][\Phi]^{-1}\{u\}_o\\ &+\frac{\Delta t_2}{t_c}[\Phi]\,[E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}}[D][\Phi]^{-1}\{u\}_o+\left(1-\frac{\Delta t_2}{t_c}\right)[\Phi]\,[E_v]^{\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [\Phi]\,[E_v]^{\frac{t_c}{2\beta}}[D][\Phi]^{-1}\{u\}_o\end{aligned} \tag{5.105}$$

Equations (5.101) to (5.105) may be rewritten as:
At point 2, the pore water pressure vector is:

$$\{u\}_2 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)} - [E_v]^{\frac{t_c}{2\beta}}\right][D][\Phi]^{-1}\{u\}_o \tag{5.106}$$

At point 3, the pore water pressure vector is:

$$\{u\}_3 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}\right)+\frac{\Delta t_2}{\beta}} - [E_v]^{\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}} + \frac{\Delta t_2}{t_c}[E_v]^{\frac{\Delta t_2}{\beta}}\right][D][\Phi]^{-1}\{u\}_o \tag{5.107}$$

At point 4, the pore water pressure vector is:

$$\{u\}_4 = [\Phi]\left[\begin{array}{l} [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} \\ + \frac{\Delta t_2}{t_c}[E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)} + \left(1-\frac{\Delta t_2}{t_c}\right)[E_v]^{\left(\frac{t_c}{2}-\Delta t_2\right)} \end{array}\right][D][\Phi]^{-1}\{u\}_o \tag{5.108}$$

At point 5, the pore water pressure vector at the end of the second cycle is:

$$\{u\}_5 = [\Phi]\left[\begin{array}{l} [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} \\ + \frac{\Delta t_2}{t_c}[E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} + \left(1-\frac{\Delta t_2}{t_c}\right)[E_v]^{\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\frac{t_c}{2\beta}} \end{array}\right][D][\Phi]^{-1}\{u\}_o \tag{5.109}$$

For simplicity equations (5.108) and (5.109) may be rewritten as:

$$\{u\}_4 = [\Phi]\left[[E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)} + [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)}\right][D][\Phi]^{-1}\{u\}_o \tag{5.110}$$

$$\{u\}_5 = [\Phi]\left[\begin{array}{l} [E_v]^{\left(\frac{t_c}{2}+\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\left(\frac{t_c}{2\beta}+\frac{\Delta t_2}{\beta}\right)+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} \\ + [E_v]^{\frac{\Delta t_2}{\beta}+\left(\frac{t_c}{2}-\Delta t_2\right)+\frac{t_c}{2\beta}} - [E_v]^{\frac{t_c}{2\beta}} \end{array}\right][D][\Phi]^{-1}\{u\}_o \tag{5.111}$$

In general, the pore water pressure vector at the end of Δt_{nc} of the n_c cycle is given by (such as point 3):

$$\{u\}_{3n_c-3} = [\Phi]\left[\frac{\Delta t_{n_c}}{t_c}[E_v]^{\frac{\Delta t_{n_c}}{\beta}} + \sum_{i=1}^{2n_c-2}(-1)^{(i+1)}[E_v]^{\left(T_i-\frac{t_c}{2\beta}-\frac{t_c}{2}+\Delta t_{n_c}\right)}\right][D][\Phi]^{-1}\{u\}_o \tag{5.112}$$

while, the pore water pressure vector at the middle of the n_c cycle is given by (such as point 4):

$$\{u\}_{3n_c-2} = [\Phi]\left[\sum_{i=1}^{2n_c-1}(-1)^{(i+1)}[E_v]^{\left(T_i-\frac{t_c}{2\beta}\right)}\right][D][\Phi]^{-1}\{u\}_o \tag{5.113}$$

and the pore water pressure vector at the end of the n_c cycle is given by (such as point 5):

$$\{u\}_{3n_c-1} = [\Phi] \left[\sum_{i=1}^{2n_c} (-1)^{(i+1)} [E_v]^{T_i} \right] [D][\Phi]^{-1} \{u\}_o \tag{5.114}$$

where:

$$T_i = \left(2n_c(1+\beta) + 2 - \sum_{k=1}^{i}\left(1-(-1)^k\right) - \beta\sum_{k=1}^{i}\left(1-(-1)^{(k+1)}\right)\right)\frac{t_c}{4\beta}$$

$$+\frac{(1-\beta)}{2\beta}\left(2\sum_{k=1}^{n_c}\Delta t_k - \sum_{k=1}^{i}\left(1-(-1)^{(k+1)}\right)\Delta t_{\left(\frac{1-(-1)^{(k+1)}}{4}\right)k}\right)$$

5.3.5.4 Formulation of pore water pressure for nonrectangular cyclic loading

Figure 5.14 shows different types of cyclic loading. The types may be represented any expected cyclic loading shape. In the figure, there are three parameters t_o, α_o and β_o reflect the properties of the loading. The number of cycle is N. The time t_o represents the time length of the subjected load whatever the load geometry is, while the time $\beta_o t_o$ is the total length of the cycle. The parameter α_o represents the load geometry, where α_o=0 creates a rectangular cyclic loading, α_o = 0.5 creates a triangular cyclic loading and $0<\alpha_o<0.5$ creates a trapezoidal cyclic loading.

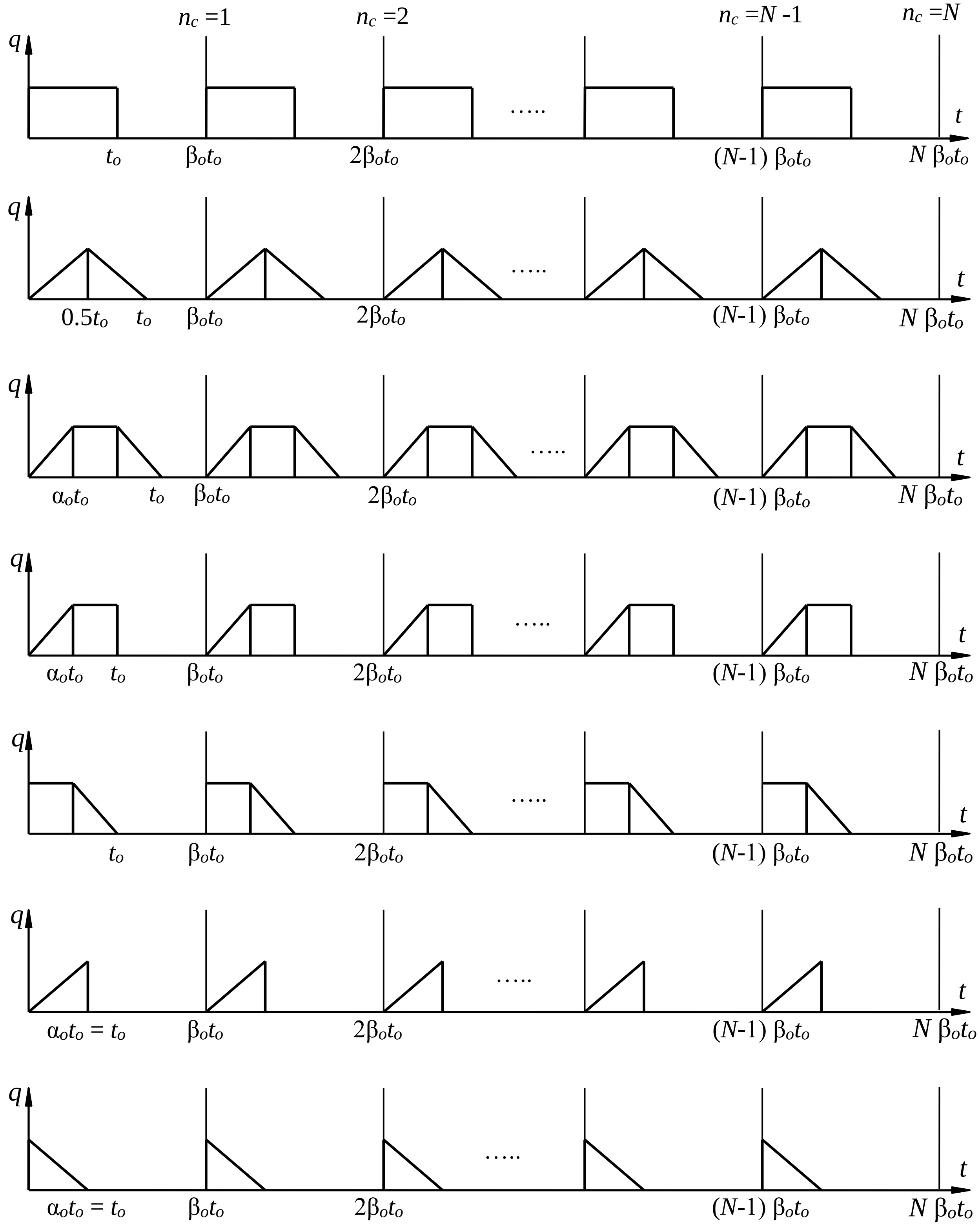


Figure 5.14   Types of cyclic loading

5.3.6 Nonlinear Analysis

Many researchers, *Zhung et al.* (2005), *Zhuang* and *Xie* (2005), *Conte* and *Troncone* (2007), *Lekha et al.* (2003), *Xie et al.* (2002, 2005 and 2006), *Chen et al.* (2005) and *Abbasi et al.* (2007), had used the assumption of nonlinear analysis proposed by *Davis* and *Raymond* (1965) to introduce a nonlinear analysis of 1-D consolidation with variable compressibility and permeability. They assumed that the decrease in permeability is proportional to the decrease in compressibility. Therefore, if the coefficient of consolidation is considered as constant during the consolidation process (Eq. 5.115), the only soil variable during the consolidation process required for the nonlinear analysis will be the coefficient of volume change.

$$C_{vi} = \frac{k_{vi}}{\gamma_w m_{vi}} = \frac{k_{voi}}{\gamma_w m_{voi}} \tag{5.115}$$

where:

γ_w Unit weight of the water, [kN/m^3]

m_{voi} Initial coefficients of volume change in a layer *i*, [m^2/kN]

k_{voi} Initial coefficients of permeability in a layer *i*, [m/year].

During the consolidation process, void ratio-effective stress response is given by:

$$e_i = e_{oi} - C_{ci} \log\left(\frac{\sigma'_i}{\sigma'_{oi}}\right) \tag{5.116}$$

where:

e_i [-] Void ratios at time *t* and the initial void ratio of layer *i* corresponding to effective stresses σ'_i [kN/m^2]

e_{oi} [-] Void ratios at time *t* and the initial void ratio of layer *i* corresponding to effective stresses σ'_{oi} [kN/m^2]

C_{ci} [-] Compressibility index of layer *i*.

From Eq. (5.116), the initial coefficient of volume change m_{voi} in a layer *i* can be estimated as follows:

$$m_{voi} = \frac{-1}{1+e_{oi}} \left.\frac{\partial e}{\partial \sigma'}\right|_{\sigma'_i = \sigma'_{oi}} = \frac{C_{ci}}{(1+e_{oi})\sigma'_{oi} \ln(10)} \tag{5.117}$$

The analysis deals with either normally or pre-consolidated clay. Therefore, the compressibility index C_{ci} [-] is replaced by the recompression index C_{ri} [-] in the case of pre-consolidated clay. Coefficient of volume change at any intermediate interval time during the consolidation process may be determined from the previous time step as described in the next sections.

5.3.6.1 Determining stresses in a soil layer

According to *Terzaghi's* principle of effective stress, the effective stress σ'_i in a layer *i* during the consolidation process can be given by:

$$\sigma'_i = \sigma'_{oi} + \Delta\sigma_i - \Delta u_i \qquad (5.118)$$

where:

σ'_{oi}	Initial vertical effective stress caused by the weight of the soil itself at the middle of layer *i*, [kN/m^2]
$\Delta\sigma_i$	Increment of vertical stress at time *t* in a layer *i* due to the applied load on the surface, [kN/m^2]
Δu_i	Average excess pore water pressure at time *t* in a layer *i*, [kN/m^2].

At the end of consolidation, $\Delta u_i = 0$ and the effective stress σ'_i in a layer *i* reaches its final value:

$$\sigma'_i = \sigma'_{oi} + \Delta\sigma_i \qquad (5.119)$$

As the load on the surface is applied gradually during the construction time, t_c and the excess pore water pressure varies during the consolidation process; the value of $\Delta\sigma_i - \Delta u_i$ defines when the clay layer is normally consolidated or pre-consolidated.

5.3.6.1.1 Normally Consolidated Clay (Loading Case)

According to Figure 5.15, a layer *i* is considered as normally consolidated clay during the consolidation process when the initial vertical effective stress σ'_{oi} is greater than or equal to the pre-consolidation pressure σ'_{ci} .

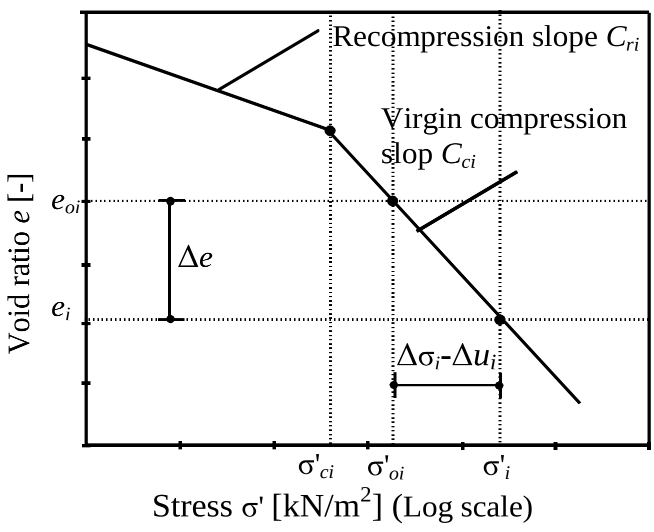


Figure 5.15 Relationship between void ratio and stress (loading case), $\sigma'_{oi} \geq \sigma'_{ci}$

For normally consolidated clay, the consolidation settlement s_{pi} of a layer i of thickness h_i is given by:

$$s_{pi} = \frac{C_{ci}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i - \Delta u_i}{\sigma'_{oi}} \right) \tag{5.120}$$

As the layer thickness is assumed to be small, the consolidation settlement s_{pi} as a function in the coefficient of volume change m_{vi} may be also given by:

$$s_{pi} = m_{vi}\left(\Delta\sigma_i - \Delta u_i \right) h_i \tag{5.121}$$

The coefficient of volume change at any time t during the consolidation process can be obtained from Eqs. (5.120) and (5.121) as follows:

$$m_{vi} = \frac{C_{ci}}{(1+e_{oi})(\Delta\sigma_i - \Delta u_i)} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i - \Delta u_i}{\sigma'_{oi}} \right) \tag{5.122}$$

5.3.6.1.2 Pre-Consolidated Clay (Reloading Case)

The reloading case for a layer i during the consolidation process is considered when the pre-consolidation pressure σ'_{ci} is greater than the effective stress σ'_i (Figure 5.16). For pre-consolidated clay (reloading case), replacing the compression index for loading C_{ci} by the compression index for reloading C_{ri}, gives the settlement s_{pi} from Eq. (5.120) and the coefficient of volume change from Eq. (5.122).

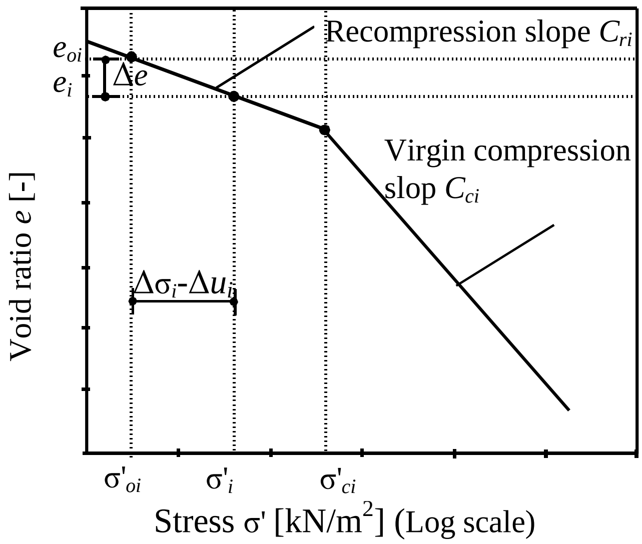


Figure 5.16 Relationship between void ratio and stress (reloading case), $\sigma'_i<\sigma'_{ci}$

5.3.6.1.3 Pre-Consolidated Clay (Case of Loading and Reloading)

In the general case of loading and reloading (Figure 5.17), the increment of vertical stress $\Delta\sigma'_i$ in a layer i is expressed as:

$$\Delta\sigma'_i = \sigma'_{ri} + \Delta\sigma'_{ei} \tag{5.123}$$

where:

$\Delta\sigma'_{ri} = \sigma'_{ci} - \sigma'_{oi}$ Reloading increment of vertical stress in a layer i, [kN/m^2]

$\Delta\sigma'_{ei} = \sigma'_{oi} + \Delta\sigma_i - \Delta u_i - \sigma'_{ci}$ Loading increment of vertical stress in a layer i, [kN/m^2].

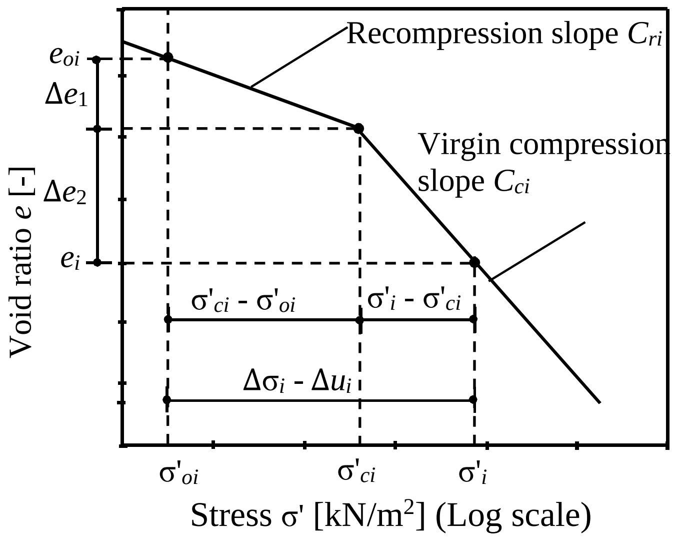


Figure 5.17 Relationship between void ratio and stress (case of loading and reloading), $\sigma'_i > \sigma'_{ci} > \sigma'_{oi}$

The total consolidation settlement is divided into two parts according to Figure 5.17. In the first part, the clay layer will consolidate with reloading compression index C_{ri} until the soil pressure reaches pre-consolidation pressure σ'_{ci}. In the second part after reaching the pressure σ'_{ci}, the clay layer will consolidate more with loading compression index C_{ci} until reaching the final stress σ'_i.

For pre-consolidated clay (the loading and reloading case), the reloading pressure effect may be taken into consideration by dividing the consolidation settlement in a layer i into two terms such that:

$$s_{pi} = \frac{C_{ri}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{ci}}{\sigma'_{oi}}\right) + \frac{C_{ci}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i - \Delta u_i}{\sigma'_{ci}}\right) \tag{5.124}$$

For the loading and reloading case, the consolidation settlement s_{pi} as a function in coefficients of volume change m_{vi} for loading and m_{ri} for reloading may be also given by:

$$s_{pi} = m_{ri}\left(\sigma'_{ci} - \sigma'_{oi}\right)h_i + m_{vi}\left(\sigma'_{oi} + \Delta\sigma_i - \Delta u_i - \sigma'_{ci}\right)h_i \tag{5.125}$$

The coefficients of volume change for loading and reloading at any time *t* during the consolidation process can be obtained from Eq. (5.124) and (5.125) as follows:

$$\left.\begin{aligned} m_{ri} &= \frac{C_{ri}}{(1+e_{oi})(\sigma'_{ci}-\sigma'_{oi})}\log\left(\frac{\sigma'_{ci}}{\sigma'_{oi}}\right) \\ m_{vi} &= \frac{C_{ci}}{(1+e_{oi})(\sigma'_{oi}+\Delta\sigma_i-\Delta u_i-\sigma'_{ci})}\log\left(\frac{\sigma'_{oi}+\Delta\sigma_i-\Delta u_i}{\sigma'_{ci}}\right) \end{aligned}\right\} \tag{5.126}$$

5.3.6.1.4 Degree of Consolidation

The degree of consolidation U_s of the entire soil layers can be expressed as:

$$U_s = \frac{S_{kt}}{S_{ku}} \tag{5.127}$$

where:

S_{kt} is the sum of primary consolidation settlements in all layers at the required time *t*, [m]

S_{ku} is the sum of final consolidation settlements in all layers, when $\Delta u_i = 0$, [m].

5.3.7 Creep Compression

Settlement does not stop even after the excess pore water pressure has completely dissipated, but continues with a slow rate under constant effective stress. This is generally known as secondary consolidation or creep. Creep can take place also simultaneously with primary consolidation. For simplicity it is generally assumed that creep occurs at the end of primary consolidation and at constant effect stress. The rate of creep can be defined by the slope $C_{\alpha i}$ of the final part of the void ratio versus log time curve of the consolidation (Figure 5.18).

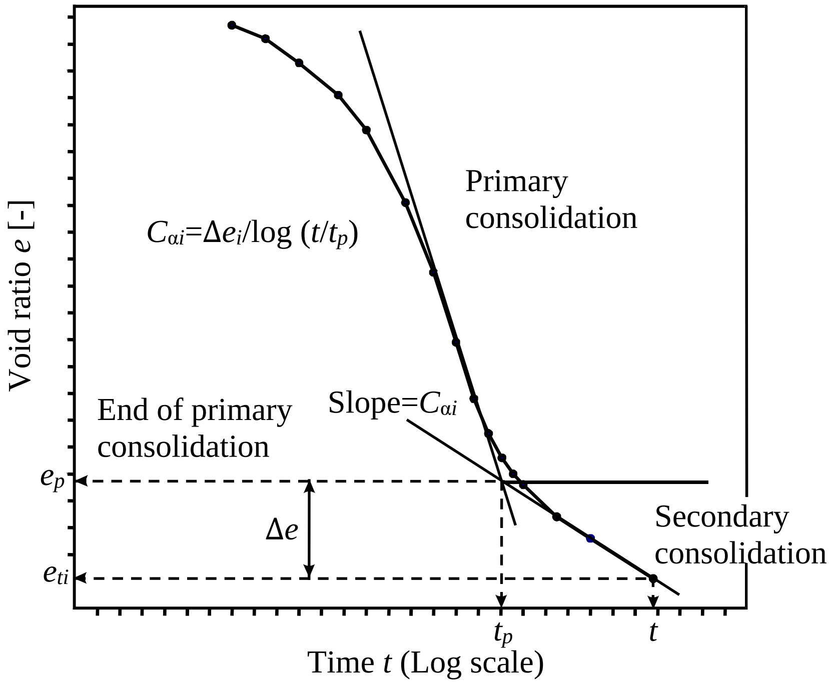


Figure 5.18 Relationship between void ratio and time

The secondary consolidation for a layer i, s_{si}, can be determined from:

$$s_{si} = \frac{C_{\alpha i} h_{pi}}{1 + e_{pi}} \log \frac{t}{t_p} \tag{5.128}$$

where:

h_{pi} Thickness of the compressible soil layer i at time t_p, [m]
t_p Time at the end of primary consolidation, [year]
t Time for which the secondary consolidation settlement is required, [year]
e_{pi} Void ratio for layer i at the end of primary consolidation, [-]
$C_{\alpha i}$ Coefficient of secondary consolidation for layer i, [-].

For normally consolidated clay, the primary consolidation settlement s_{pi} of a layer i of thickness h_i just after excess pore water pressure has completely dissipated maybe written as:

$$s_{pi} = \frac{e_{oi} - e_{pi}}{1 + e_{oi}} h_i \tag{5.129}$$

Equating Eq. (5.128) to Eq. (5.120) with eliminating Δu_i in Eq. (5.120) where the secondarily consolidation begins after excess pore water pressure has completely dissipated, gives the void ratio at the end of primary consolidation by:

$$e_{pi} = e_{oi} - C_{ci} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right) \tag{5.130}$$

while the thickness of the compressible soil layer at the end of primary consolidation, h_{pi}, is given by:

$$h_{pi} = h_i - s_{pi} = h_i - \frac{C_{ci}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right) \tag{5.131}$$

Substituting Eq. (5.130) and Eq. (5.131) into Eq. (5.129), gives the secondary consolidation, s_{si} for a layer i by:

$$s_{si} = \frac{C_{\alpha i}\left(h_i - \frac{C_{ci}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right)\right)}{1 + e_{oi} - C_{ci} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right)} \log \frac{t}{t_p} \tag{5.132}$$

Similarly, the secondary consolidation for the reloading case is given by:

$$s_{si} = \frac{C_{\alpha ri}\left(h_i - \frac{C_{ri}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right)\right)}{1 + e_{oi} - C_{ri} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{oi}} \right)} \log \frac{t}{t_p} \tag{5.133}$$

while that for the loading and reloading case is given by:

$$s_{si} = \frac{C_{\alpha ri}\left(h_i - \frac{C_{ri}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{ci}}{\sigma'_{oi}} \right) - \frac{C_{ci}\, h_i}{1+e_{oi}} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{ci}} \right)\right)}{1 + e_{oi} - C_{ri} \log\left(\frac{\sigma'_{ci}}{\sigma'_{oi}} \right) - C_{ci} \log\left(\frac{\sigma'_{oi} + \Delta\sigma_i}{\sigma'_{ci}} \right)} \log \frac{t}{t_p} \tag{5.134}$$

where $C_{\alpha ri}$ is the reloading coefficient of secondary consolidation for layer i, [-].

In Eqs. (5.132) to (5.134) the time at the end of primary consolidation t_p needs to be estimated from the degree of consolidation. The time at the end of primary consolidation t_p is considered when the excess pore water pressure $u_i(z, t)$ is less than a small value ε. This value may be taken as $\varepsilon = 10^{-8}$.

5.3.8 Application results

A user-friendly computer program has been developed for solving time-dependent problems of clay layers using the method outlined in this book. With the help of this program, an analysis of a general time-dependent problem was evaluated to show the behavior of excess pore water pressure during the consolidation process taking into account changing the state from over- to normally-consolidated clay.

5.3.8.1 Nonlinear Analysis of a Square Foundation on A Finite Thick Clay Layer

An application of *LEM* is carried out to study the behavior of a foundation resting on a thick clay layer. A square foundation shape is considered with a side of B = 10 [m]. The foundation is subjected to a uniform load of q_o = 100 [kN/m^2] and rests on a clay layer of thickness H = 2B as shown in Figure 5.19. The groundwater level lies at a depth equal to the foundation depth D_f.

The study examines effects of reloading contact pressure on the nonlinear consolidation behavior. In the example, the total contact pressure on the raft is divided into two terms. The first term is reloading contact pressure $q_v=\gamma D_f$ and the second term is loading contact pressure $q_e= q_o-q_v$. Consequently, the total settlement is obtained from two parts. The first part is due to the reloading contact pressure q_v, which is estimated from C_r; and the second part is due to the loading contact pressure q_e, which is estimated from C_c. Sequence of unloading, loading and reloading are not considered.

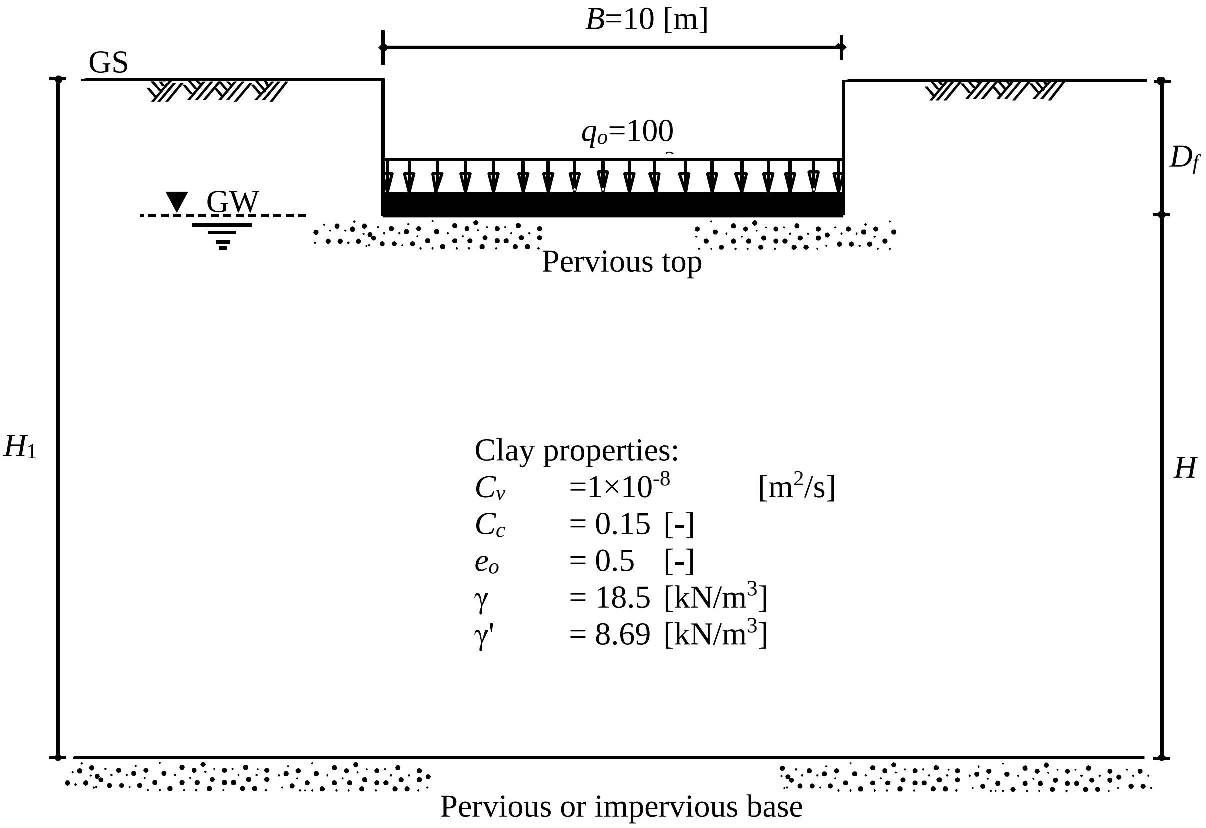


Figure 5.19 Section in clay layer with dimension and soil properties

5.3.8.1.1 Clay properties

The compression indices for loading C_c and for reloading C_r along with the initial void ratio e_o are used to define the consolidation characteristics of the clay. For the comparison purpose, the compression index C_c and the initial void ratio e_o are chosen to make the term $C_c/(1+e_o)$ as a constant and equal to 0.1 for all calculations. The compression index for reloading C_r is assumed such that a compression index ratio, $r_c = C_c/C_r$, gives values 1 [-], 5 [-], 10 [-] and 15 [-]. The clay properties in this case are:

Coefficient of consolidation	C_v	$= 1\times10^{-8}$	[m²/s]
Compression index for loading	C_c	= 0.15	[-]
Initial void ratio	e_o	= 0.5	[-]
Unit weight of the clay	γ	= 18.5	[kN/m³]

5.3.8.1.2 Variables

Referring to Figure 5.19, the foundation depth D_f is chosen to give different values of reloading contact pressure q_v. Values of foundation depth and their corresponding variables are shown in Table 5.1.

Table 5.1 Values of foundation depth and their corresponding variables

Foundation depth D_f [m]	0.00	1.35	2.70	4.05	5.41
Total clay layer H_1 [m]	20.00	21.35	22.70	24.05	25.41

Reloading contact pressure $q_v = \gamma\ D_f$ [kN/m^2]	0	25	50	75	100
Reloading pressure ratio $r_q = q_v / q_o$ [-]	0	0.25	0.50	0.75	1.0

To carry out the analysis, the clay layer of thickness H under the foundation level is subdivided into 10 equal sub-layers with 11 nodes, each of thickness 2.0 [m], while the consolidation time is divided into 10 equal intervals. The pre-consolidation pressure in a node i on the clay layer is given by:

$$\sigma'_{ci} = \sigma'_{oi} + \Delta\sigma_{vi} \tag{5.135}$$

where $\Delta\sigma_{vi}$ is the increase of vertical stress due to the reloading pressure at node i, [kN/m^2].

5.3.8.1.3 Analysis and Results

To achieve the study, 592 computational analyses have been carried out for the above variables and parameters. To get general relations between the different variables, results are plotted in dimensionless ratios. The reloading effect is expressed by a reloading pressure ratio r_q [-], which is given by:

$$r_q = \frac{q_v}{q_o} \tag{5.136}$$

where:

q_o Contact pressure on the foundation, [kN/m^2]

$q_v = \gamma\ D_f$ Reloading contact pressure, [kN/m^2].

Similarly, the settlement effect is expressed by a settlement ratio r_s [-], which is given by:

$$r_s = \frac{S_v}{S_o} \tag{5.137}$$

where:

S_o Central settlement due to the applied load without reloading effect ($r_q = 0$), [m]

S_v Central settlement due to the applied load with reloading effect ($r_q > 0$), [m].

Figure 5.20 and Figure 5.21 show comparisons between U_p and U_s for pervious and impervious boundaries at different values of T_v when $r_c = 1$, $r_c = 10$ and $r_q = 0.5$. Unlike the linear analysis where U_p is equal to U_s, the figures indicate that U_p and U_s are not equal for either $r_c = 1$ or $r_c = 10$. The development of settlement in the case of $r_c = 1$ is quicker than the dissipation of excess pore water pressure (i.e., $U_s > U_p$). This observation is changed in the case of $r_c = 10$. The greatest rate of consolidation occurs for a pervious boundary.

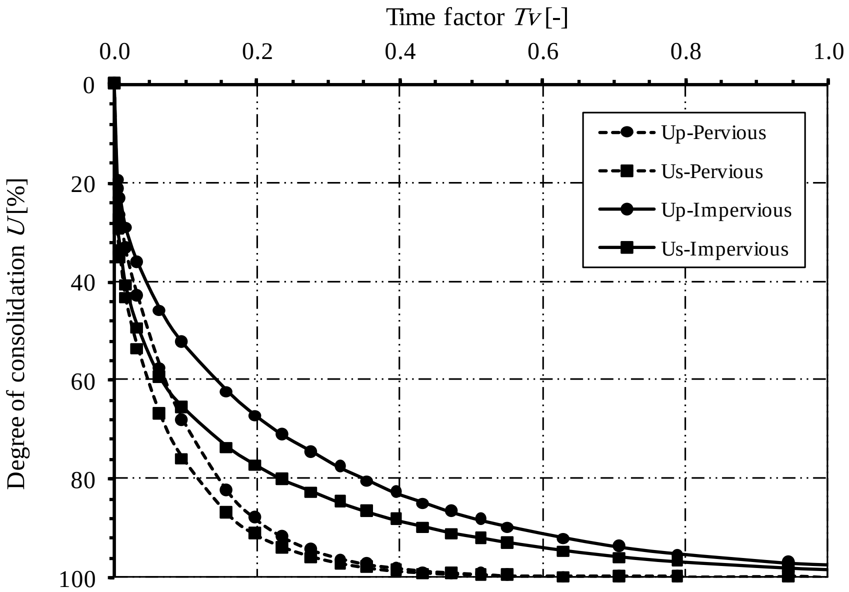


Figure 5.20 Comparison between U_p and U_s (r_c=1 and r_q=0.5)

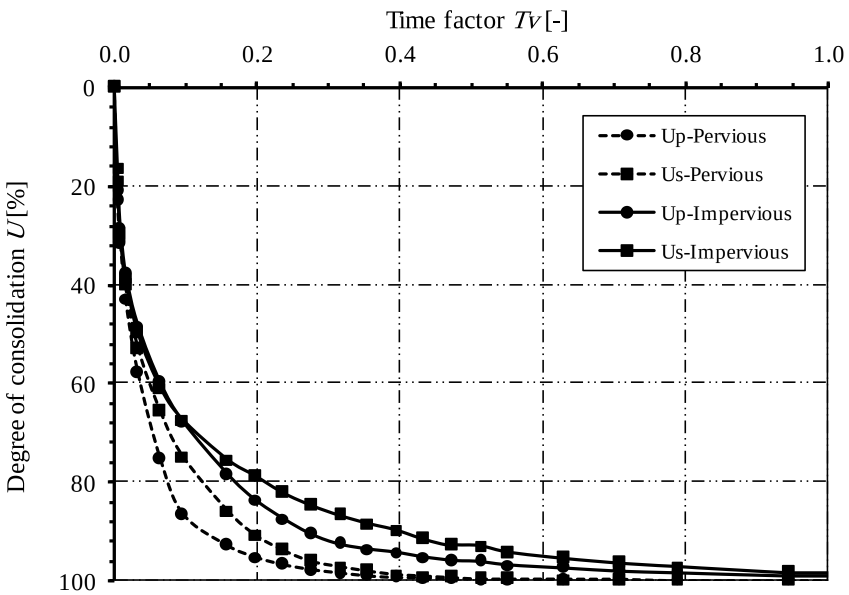


Figure 5.21 Comparison between U_p and U_s (r_c=10 and r_q=0.5)

Figure 5.22 to Figure 5.25 show the variations in settlement ratio r_s with the reloading pressure ratio r_q for r_c = 1 and r_c = 10 at different values of T_v.

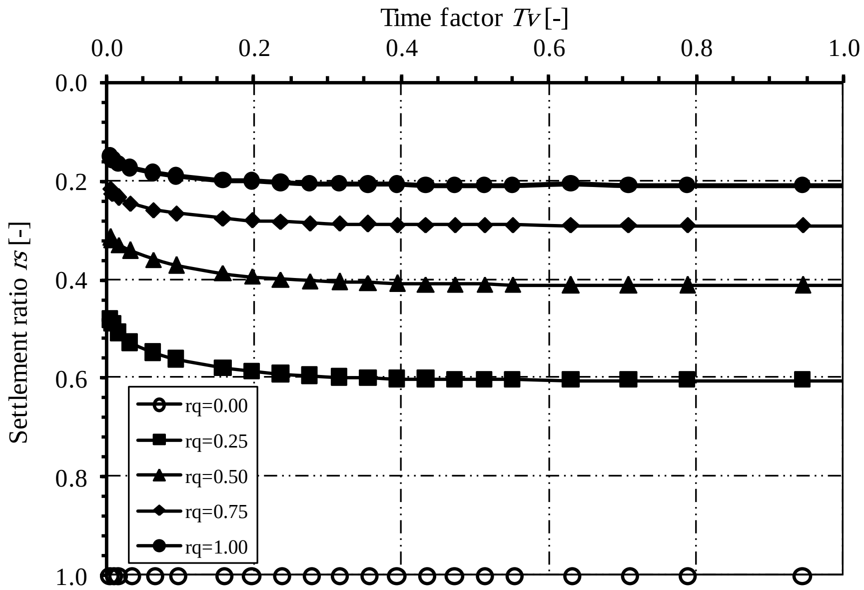


Figure 5.22 Effect of reloading pressure on the settlement ratio r_s (pervious boundary-r_c=1)

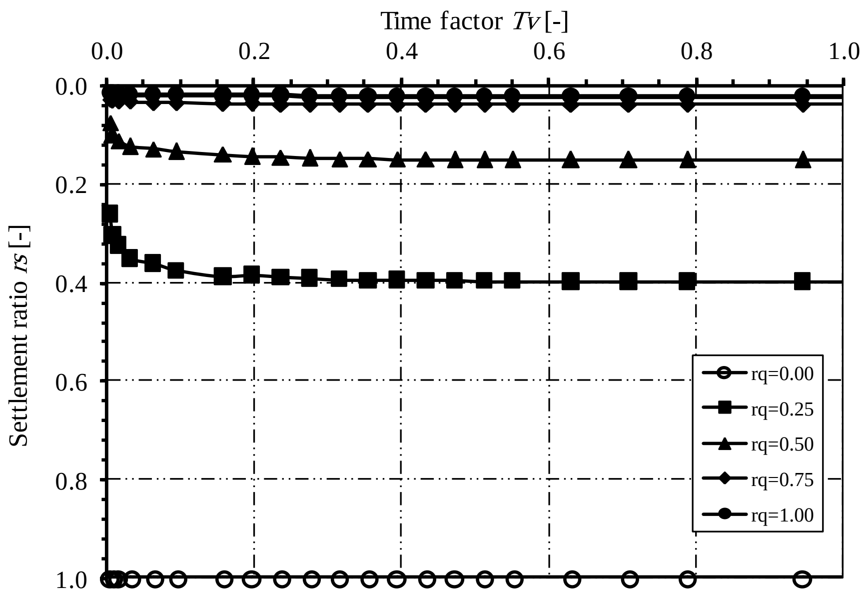


Figure 5.23 Effect of reloading pressure on the settlement ratio r_s (pervious boundary-r_c=10)

It can be observed from these figures that the reloading pressure has a considerable influence on the settlement. To further clarify, consider the following three values of reloading pressures; r_q =0.75, r_q = 0.25 and r_q = 0.0 for a pervious boundary at consolidation time of t=40 years (T_v = 0.0315) and r_c = 10, which may represent relatively large and small values of r_q. Results of these three cases are presented in values in Table 5.2. Referring to this table, as a sample, r_q = 0.75 meets a foundation depth of D_f = 4.05 [m], which gives a reloading contact pressure of q_v = 75 [kN/m^2]. This means that the first term of settlement is determined from reloading contact pressure of q_v = 75 [kN/m^2] and Compression index for reloading C_r = 0.015 [-]. In this case, the second term of settlement is determined from loading contact pressure of $q_e = q_o - q_v$ = 25 [kN/m^2] and Compression index for loading C_r = 0.15 [-]. Similarly, the settlement terms for the other two cases can be determined.

Table 5.2 Values of results at consolidation time of 40 years

Foundation depth D_f [m]	0.00	1.35	2.70	4.05	5.41
Total clay layer H_1 [m]	20.00	21.35	22.70	24.05	25.41
Reloading contact pressure $q_v = \gamma\ D_f$ [kN/m^2]	0	25	50	75	100
Reloading pressure ratio $r_q = q_v / q_o$ [-]	0	0.25	0.50	0.75	1.0
Total settlement S_v [cm]	37.68	13.22	4.66	1.24	0.66
Settlement ratio $r_s = S_v/S_o$ [-]	1.00	0.35	0.12	0.03	0.02

From Table 5.2, it can be concluded that at a relatively small value of r_q = 25, the settlement reduces to 35 [%] of that without reloading pressure (at r_q = 0 and D_f = 0), while at a relatively greater value of r_q =75, the settlement reduces to 12 [%] of that without reloading pressure. These percentages become 51 [%] and 25 [%] when r_c = 1.

The development of settling begins quickly and becomes constant until the end of consolidation process. Both two cases of pervious and impervious boundaries give nearly the same settlement magnitude at a specified r_c.

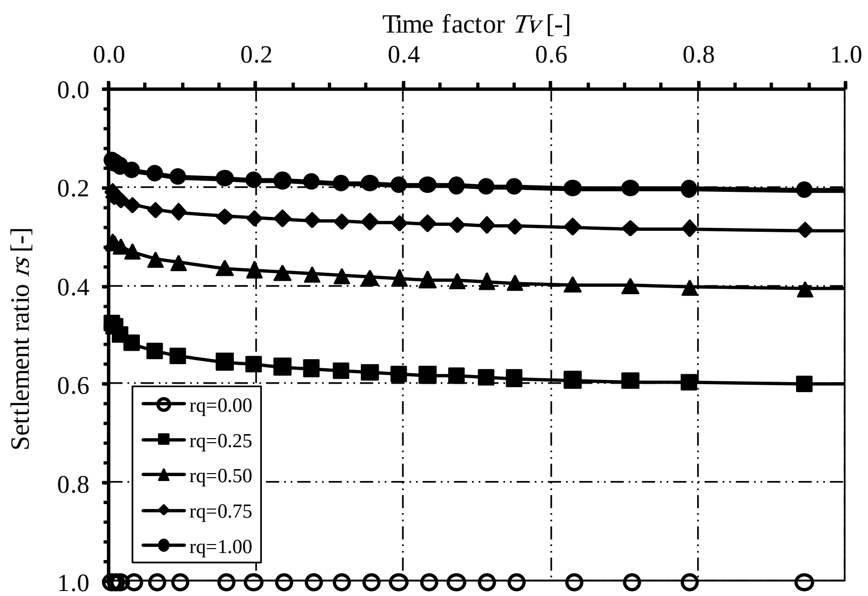


Figure 5.24 Effect of reloading pressure on the settlement ratio r_s (impervious boundary-r_c=1)

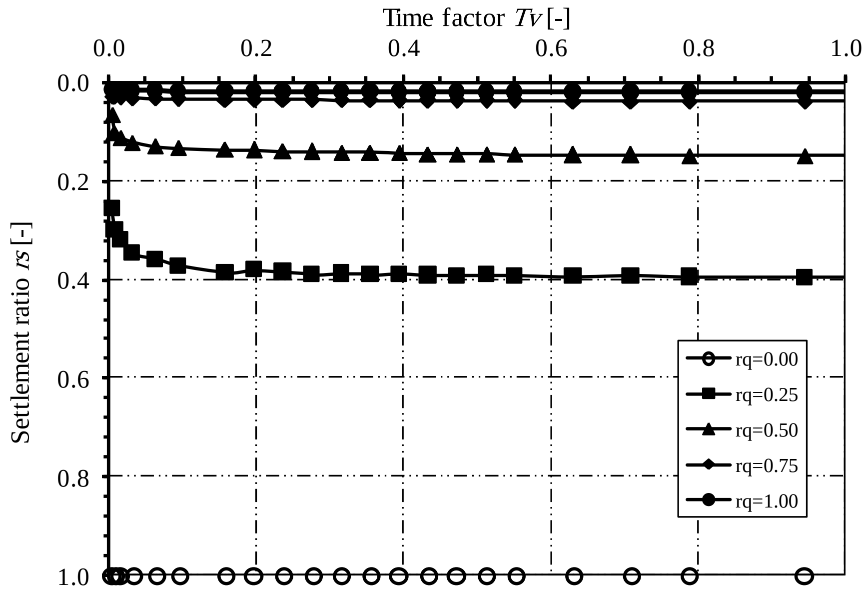


Figure 5.25 Effect of reloading pressure on the settlement ratio r_s (impervious boundary-r_c=10)

The effect of the compression index ratio r_c on the degree of consolidation U_s at different values of T_v when r_q=0.5 for pervious and impervious boundaries are shown in Figure 5.26 and Figure 5.27. The figures show that the rate of consolidation for a pervious boundary is higher than that for an impervious boundary. The rate of consolidation U_s for all values of r_c are nearly the same for either pervious or impervious boundary.

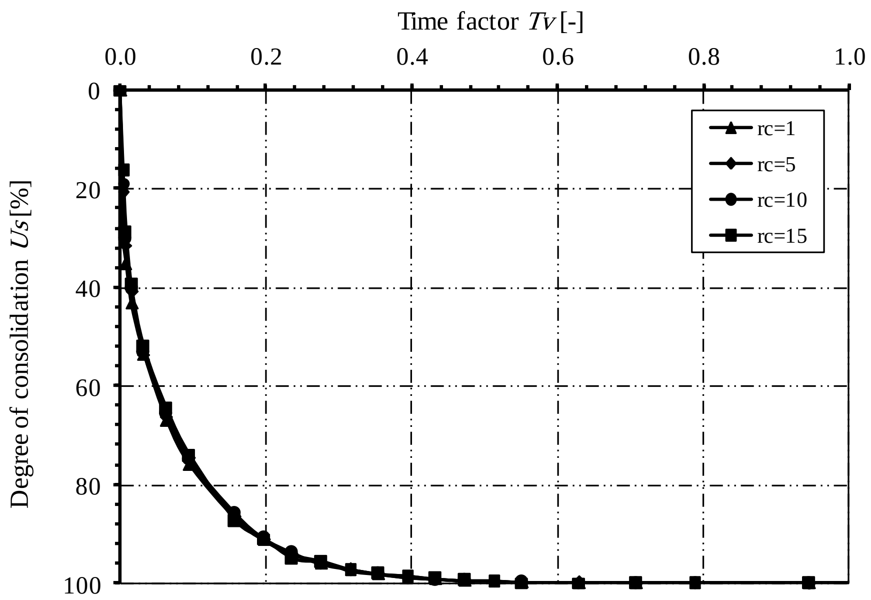


Figure 5.26 Effect of index ratio on the degree of consolidation U_s (pervious boundary-r_q=0.5)

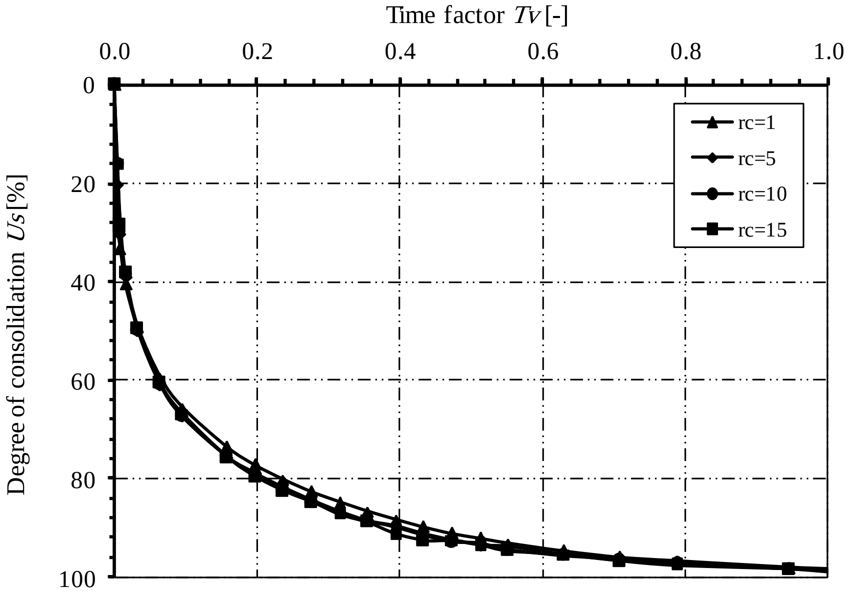


Figure 5.27 Effect of index ratio on the degree of consolidation U_s (impervious boundary-r_q=0.5)

Figure 5.28 shows the effect of loading rate on the degree of consolidation U_s at different values of T_v and $T_{vc} = C_v\, t_c/H^2$ when r_c=10 and r_q=0.5 an for pervious boundary, while Figure 5.29 shows this effect for an impervious boundary. As expected, the quicker the loading the faster the consolidation.

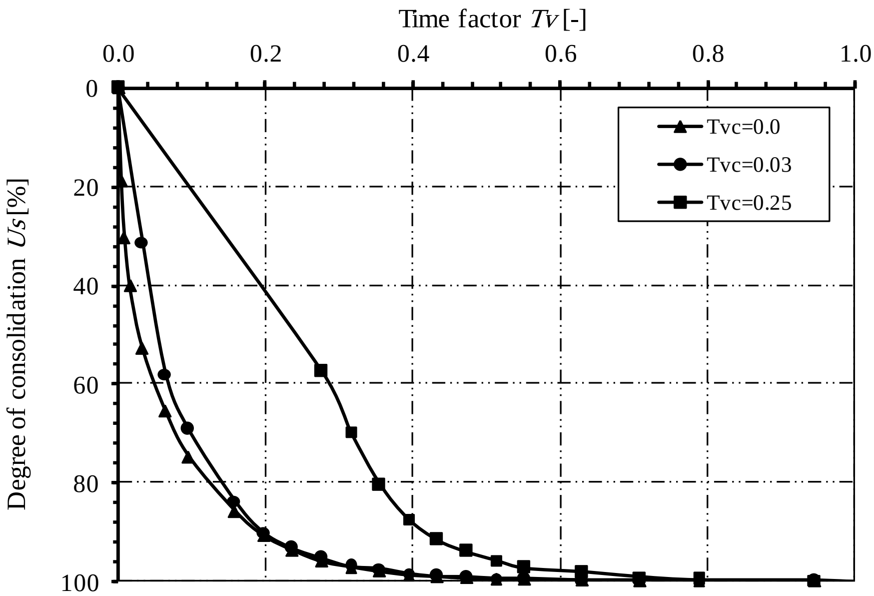


Figure 5.28 Effect of loading rate on the degree of consolidation U_s
(pervious boundary r_c=10-r_q=0.5)

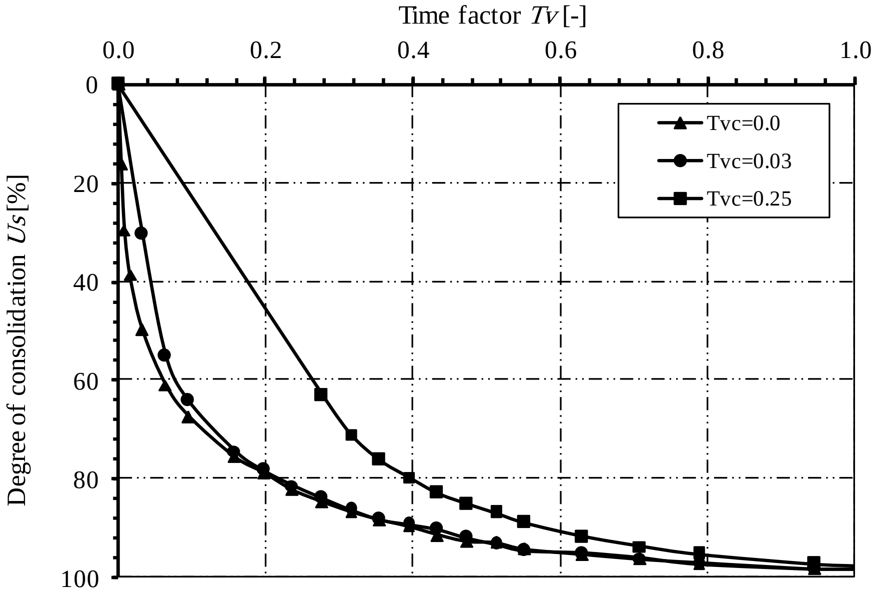


Figure 5.29 Effect of loading rate on the degree of consolidation U_s (impervious boundary r_c=10-r_q=0.5)

5.3.9 Stability of the Analysis

The solution of *LEM* is considered to be convergence when the differential equation operator λ_j of a problem can be obtained. The stability of the solution is investigated by choosing different studying nodes with different element thicknesses of clay. It is found that the solution for nonlinear analysis depends on the sub-layer thickness in which a thickness of 2 [m] gives a good results. For linear analysis the Test Example 8 in section 5.5.9 is studied. The solution was always convergence even for a very small sub-layer thickness of 0.05 [m].

For case of a system having too many layers with extreme differences in soil properties, the convergence of the solution may be not occurred. It can overcome this problem by choosing sub-layers lead to the parameter of the coefficient of consolidation and thickness, $\mu_i = (h_i / h_1)\sqrt{c_{v1}/c_{vi}}$ for all layers nearly equal to 1. In *GEO Tools*, a system of sub-layer thickness and number is generated automatically in order to let the solution to be stable.

5.4 Defining the project data

5.4.1 Firm Header

When printing the results, the main data (firm name) are displayed on each page at the top in two lines or in graphic presentation at the identification box. Firm name can be defined, modified and saved using the "Firm Header" Option from the setting Tab (see Figure 5.30).

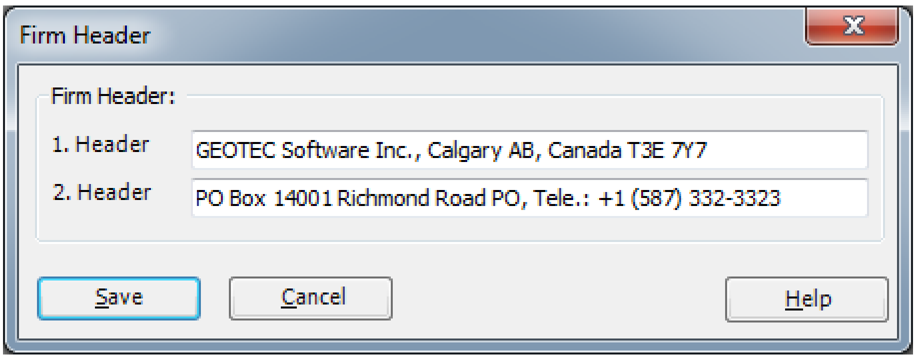


Figure 5.30 Firm Header

5.4.2 Task of the program *GEO Tools* (Analysis Type)

The program *GEO Tools* can be used to analyze various problems in Geotechnical Engineering for shallow foundations and deep foundations, Figure 5.31.

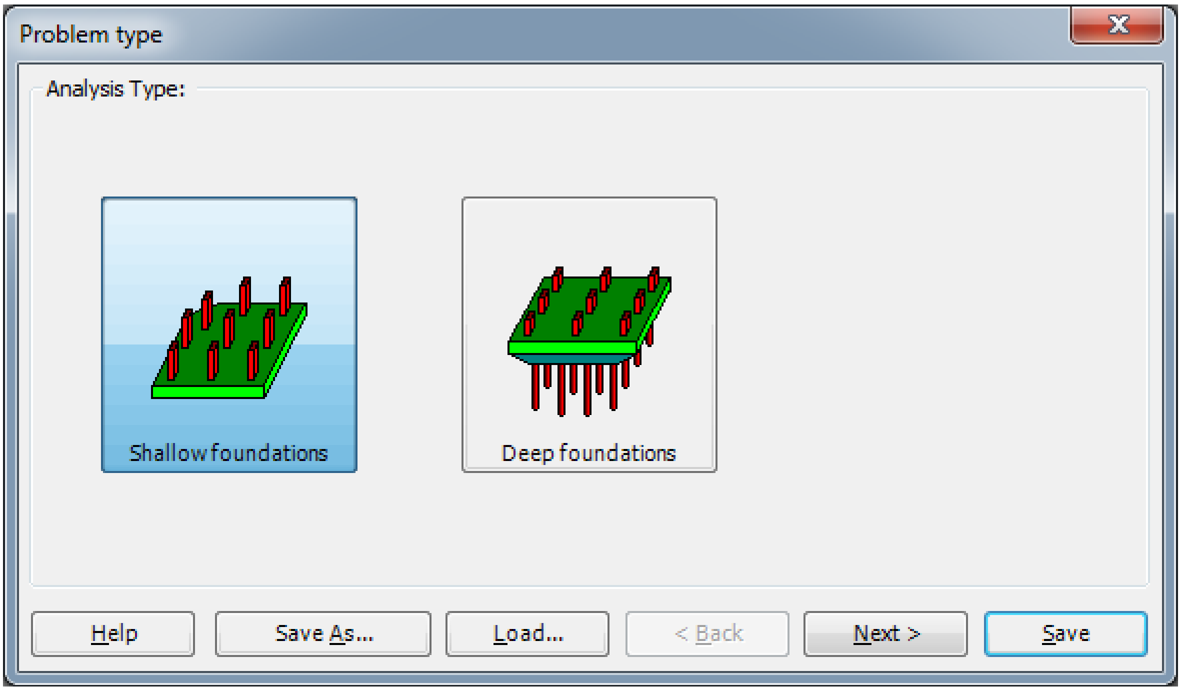


Figure 5.31 Problem type

According to the main menu (Figure 5.32) the following geotechnical problems can be calculated for shallow foundations:

1. Stresses in soil
2. Strains in soil
3. Displacements in soil
4. Consolidation settlement
5. Degree of consolidation
6. Time-settlement curve
7. Displacements of rigid raft
8. Consolidation of rigid raft
9. Settlements of footing groups

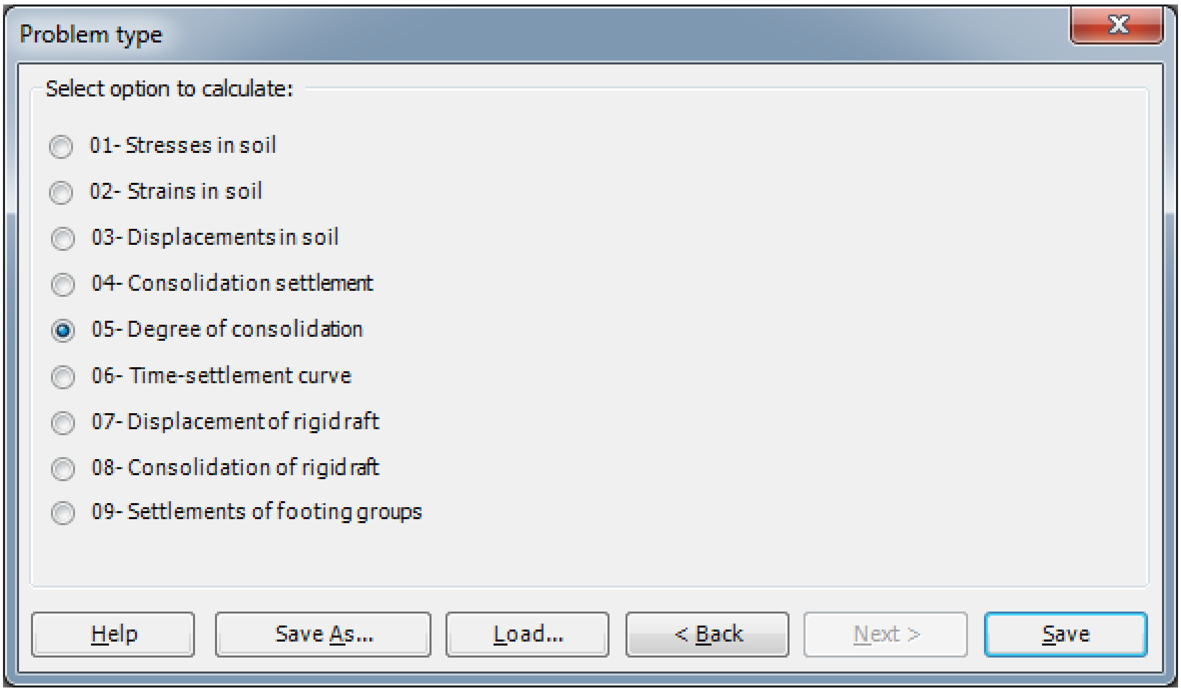


Figure 5.32 Problem type for deep foundation

In menu of Figure 5.32 select the option:

05- Degree of consolidation

The following paragraph describes how to analyze a problem of time-dependent consolidation of clay by the program *GEO Tools*. The input data are the type of loading, time of consolidation and the properties of the soil layers.

5.4.3 Project Identification

In the program, it must be distinguished between the following two data groups:

1. System data (For identification of the project that is created and information to the output for the printer).
2. Soil data (Soil properties and so on).

The defining input data for these data groups is carried out as follows:

After clicking on the "Project Identification" Option, the following general project data are defined (Figure 5.33):

Title:	Title label
Date:	Date
Project:	Project label

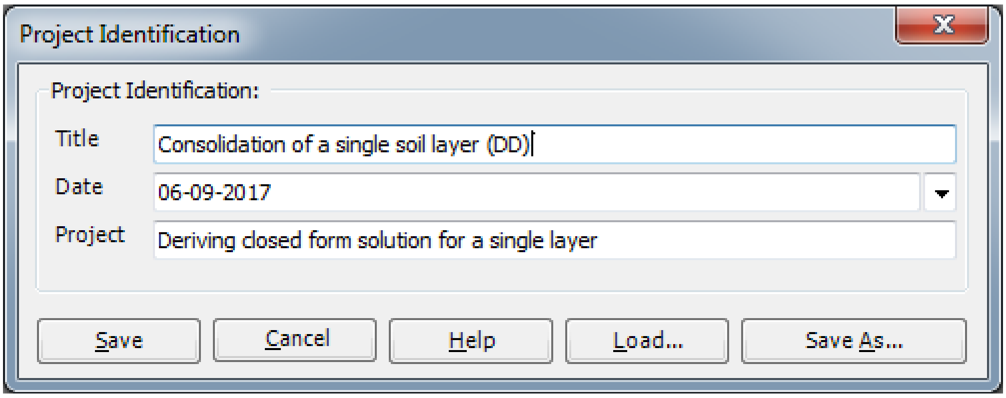


Figure 5.33 Project Identification

5.4.4 Data of degree of consolidation

After clicking on the "Degree of consolidation" Option, the following data of the consolidation problem are defined (Figure 5.34):

Calculation Task:
- Linear Analysis
- Nonlinear Analysis

Method:
- Layer Equation Method (LEM)
- Finite Difference Method (LEM)
- Eigenvalue Method (EVM)

Loading Type:
- Constant loading
- Linear Loading
- Cyclic Loading

Time:

-	Tr	Time of consolidation	[Years]
-	Tc	Time of construction	[Years]
-	T1		[Years]
-	T2		[Years]
-	T3		[Years]
-	T4		[Years]
-	dt	Time increment	[Years]

- Np No. of Periods [-]

Drainage condition:

- Impervious bottom boundary
- Pervious bottom boundary

Time Unit:

- [Years]

Generation of time

- To Start time [Years]

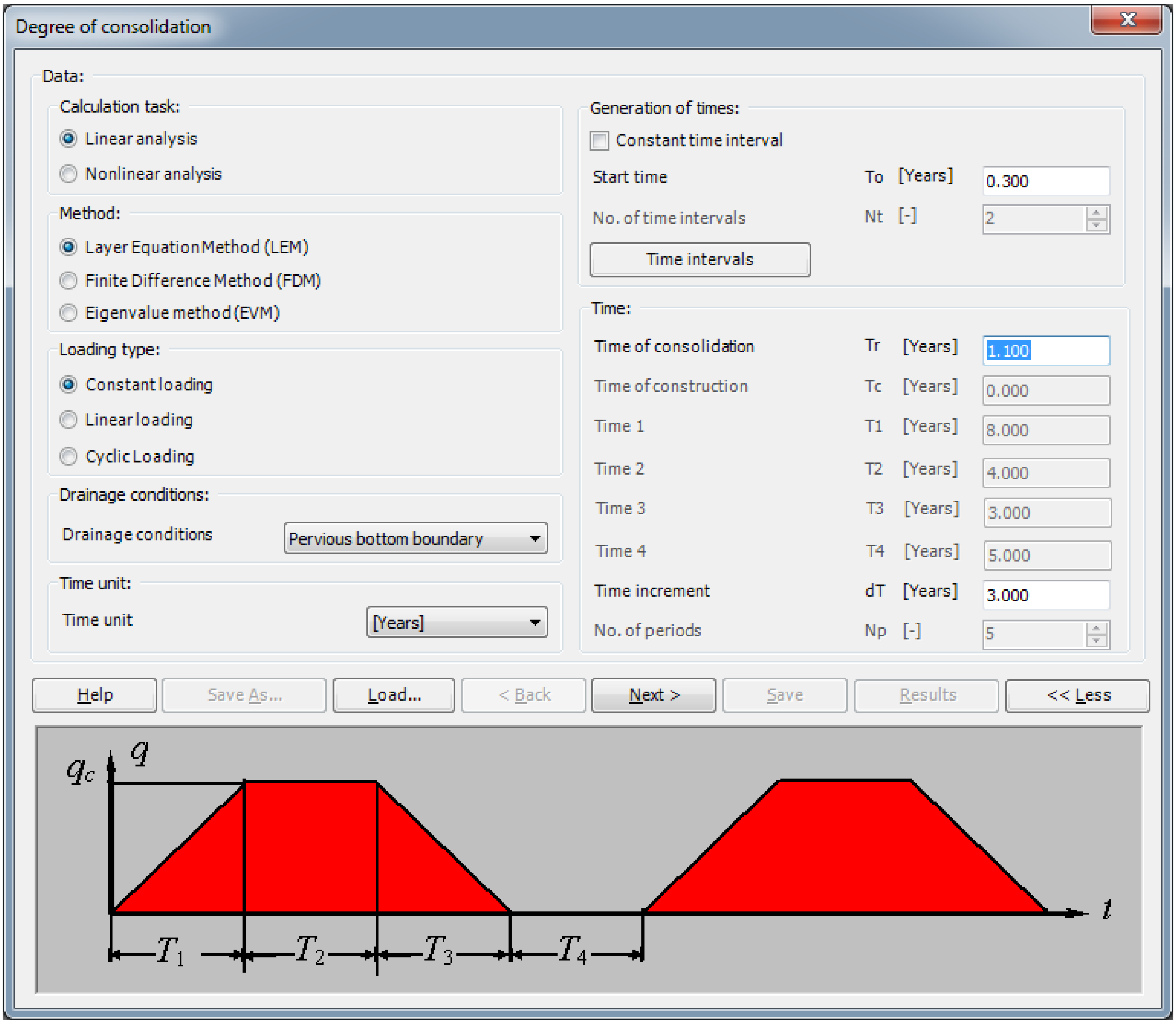


Figure 5.34 Data of the degree of consolidation

5.4.5 Data of soil layers

After clicking on "Next" button in the form of Figure 5.34, the following soil properties of the clay layers are required to define (Figure 5.35):

- Layer thickness *h* [m]
- Coefficient of consolidation C_v [m^2/s]
- Coefficient of permeability *k* [m/s]

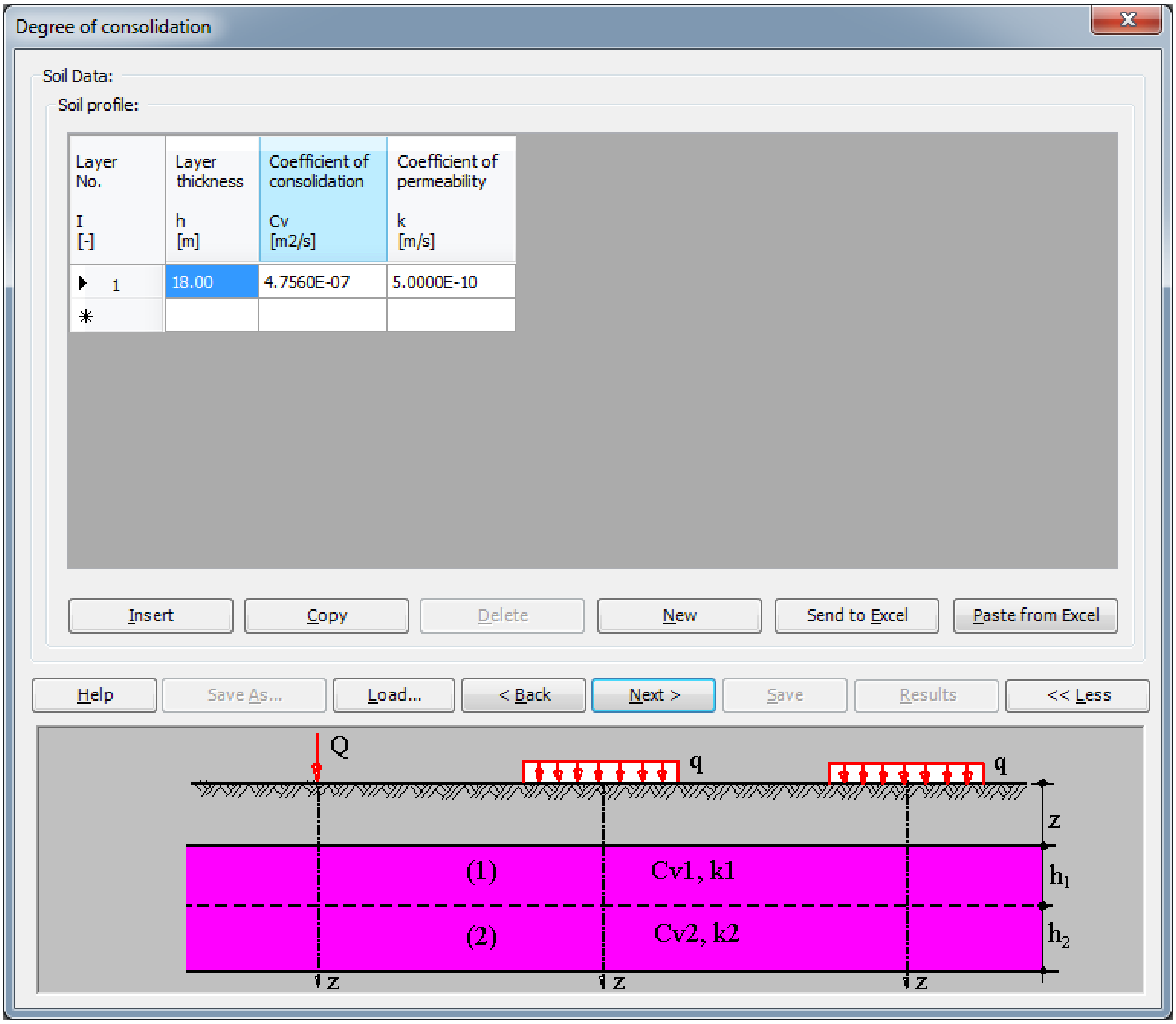


Figure 5.35 Soil data

5.4.6 Pore water pressure is

After clicking on "Next" button in the form of Figure 5.35, the following pore water pressure data are required to define (Figure 5.36):

- Constant Pore Water Pressure u_o [kN/ m^2]
- Load geometry
- Load coordinates/Layers
- Depth increment in z- Direction D_i [m]
- β for nonlinear analysis
- α for nonlinear analysis

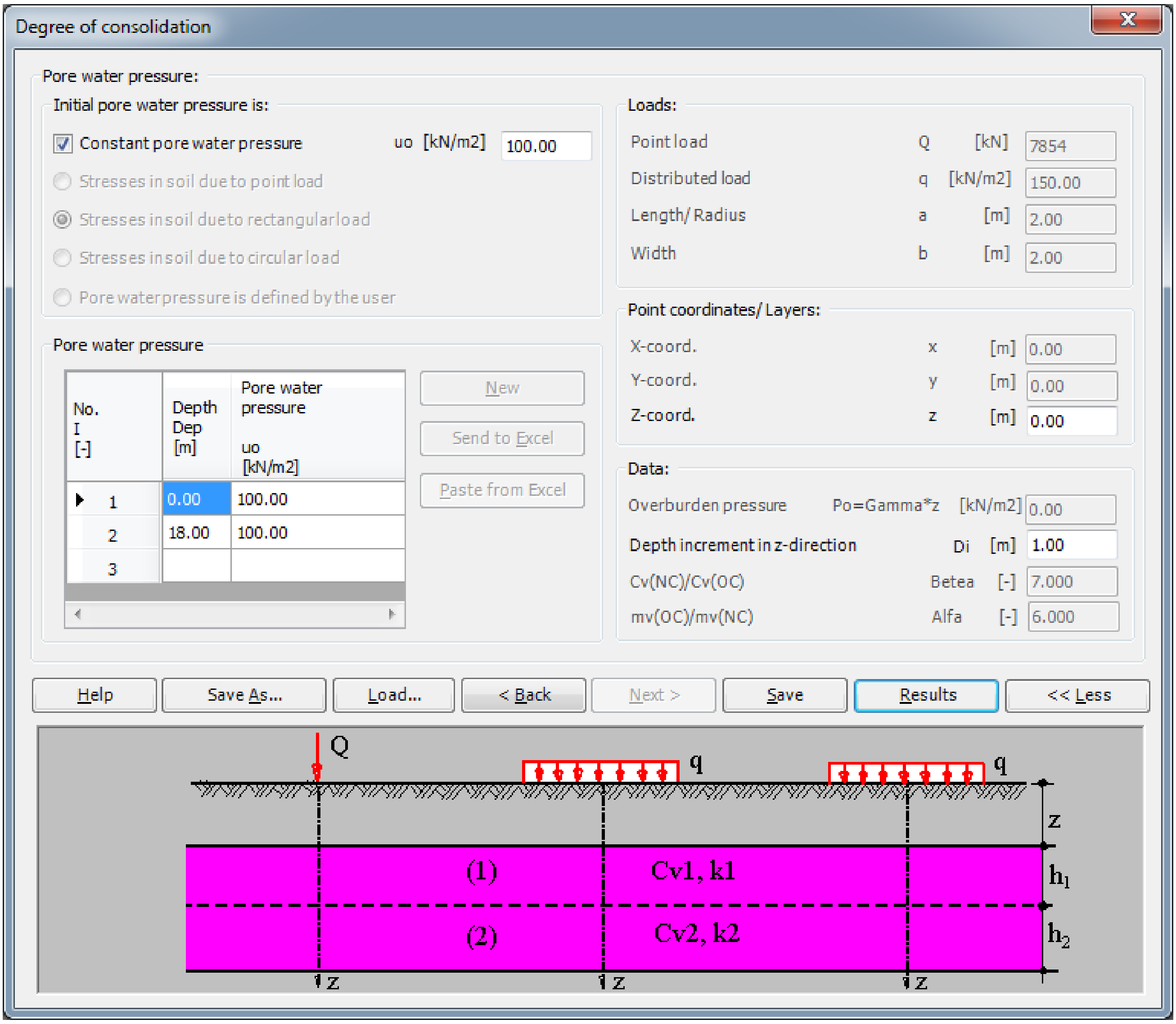


Figure 5.36 Soil data

After finishing from defining the input data, click on "Save" button to save the data then click on "Result" button to carry out the calculation and see the results.

5.5 Examples to Verify Consolidation Rate under Constant Loading

5.5.1 Introduction

A user-friendly computer program *GEO Tools* has been developed for solving time-dependent settlement problems of clay layers using three different numerical methods:

- *L*ayer *E*quation *M*ethod (*LEM*), that was developed by *Herrmann*/ *El Gendy* (2014).
- *F*inite *D*ifference *M*ethod (*FDM*), that is the traditional solution of consolidation problems.
- *E*igen *V*alue *M*ethod *(EVM*).

With the help of this program, an analysis of different examples was carried out to verify and test the methods and the program for analyzing 1-D consolidation problems.

5.5.2 Example 1: Consolidation of a Single Soil Layer by *FDM*

5.5.2.1 Description of the problem

To verify *FDM* in *GEO Tools* for time-dependent settlement problems, the excess pore water pressure for a single clay layer calculated numerically using *FDM* by *Craig* (2007), Example 7.6, page 209, is compared with that obtained by *GEO Tools*.

A clay layer of 10 [m] thick and coefficient of consolidation C_v = 7.9 [m^2/year] is considered. The layer has an impermeable bottom boundary. The initial distribution of excess pore water pressure is listed in Table 5.3 and Figure 5.37. It is required to determine the value of excess pore water pressure after consolidation has been in progress for one year.

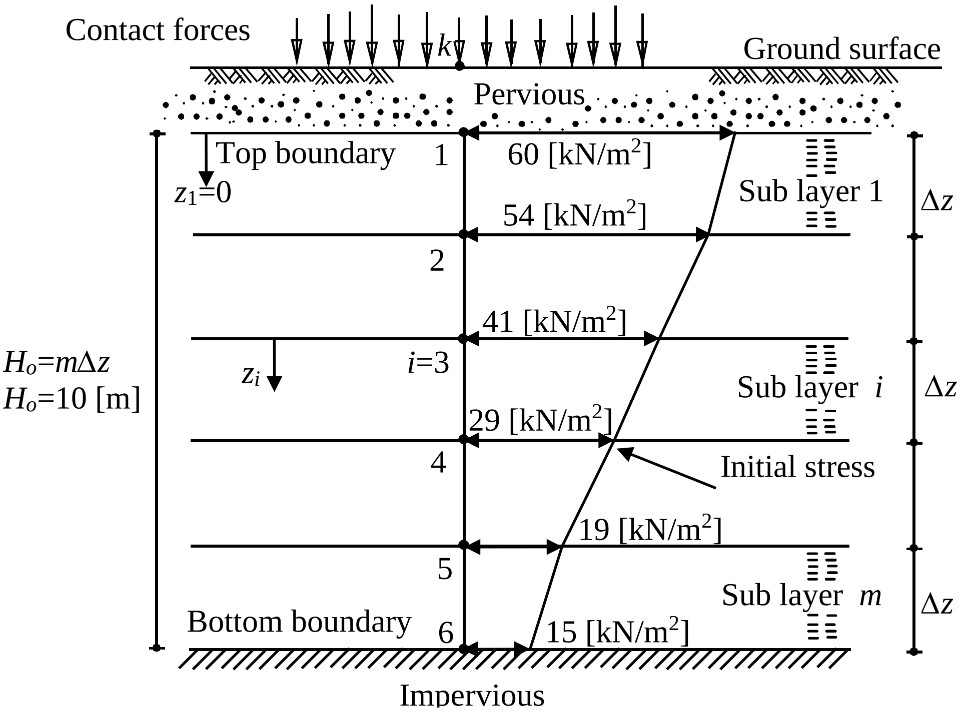


Figure 5.37 Initial excess pore water pressure on the clay layer

Table 5.3 Initial distribution of excess pore water pressure

Depth [m]	0	2	4	6	8	10
Pressure [kN/m^2]	60	54	41	29	19	15

5.5.2.2 Analysis of the problem

Craig (2007) has chosen 5 mesh intervals in the vertical direction for his analysis. To create a variable initial distribution of excess pore water pressure on the clay layer in *GEO Tools*, the whole layer is defined as 5 layers each of 2 [m] thickness and has the same properties. Then, *GEO Tools* subdivides automatically each small layer of 2 [m] thickness to the required grid nodes for the analysis. For pervious top boundary, the initial excess pore water pressure at the clay surface will tend to zero, $u_{top} = 0$.

5.5.2.3 Results

Results by *GEO Tools* are compared with those obtained by *Craig* (2007). Figure 5.38 shows excess pore water pressure with depth after one year. The values of excess pore water pressure are in a good agreement with those of *Craig* (2007). The input data and results of *GEO Tools* for this example are presented also on the next pages.

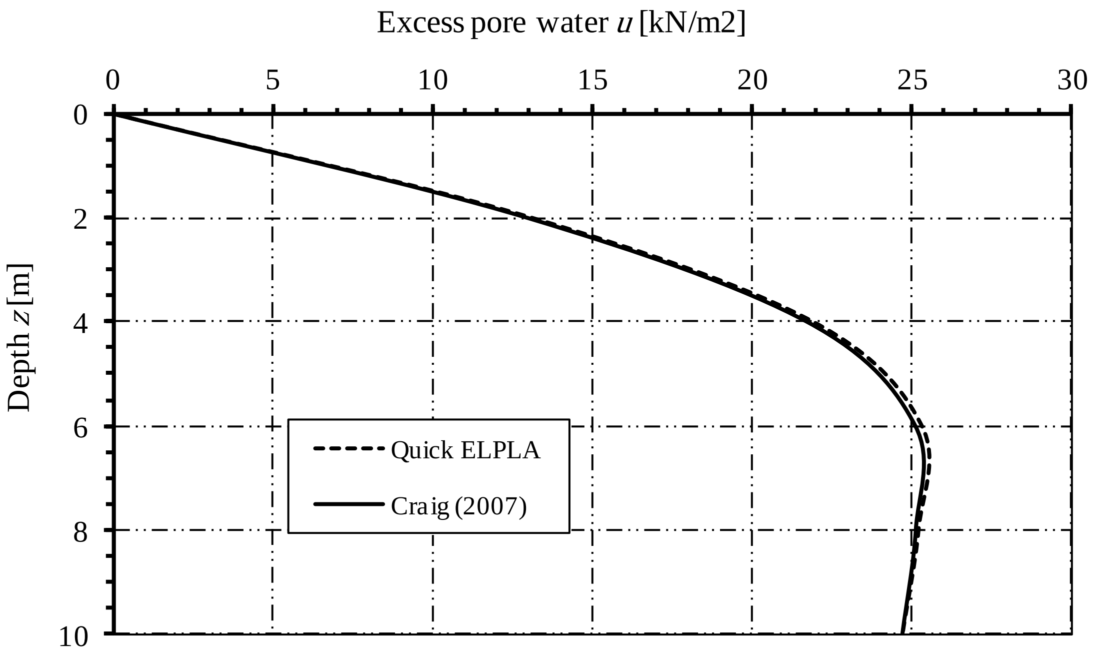


Figure 5.38 Excess pore water pressure with depth after one year

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a single soil layer (SD)
Date: 13-09-2017
Project: Craig (2007), Example 7.6, page 209
File: Craig (2007)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Pore water pressure is defined by the user
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00
```

Point coordinates/ Layers:

Layer thickness	Hb	[m]	= 10.00
Depth increment in z-direction	Di	[m]	= 2.00

Time:

Time of consolidation	Tr	[Years]	= 1.000
Time increment	dT	[Years]	= 0.100

Generation of times:

Start time	To	[Years]	= 0.000
No. of time intervals	Nt	[-]	= 10
Time interval	Ti	[Years]	= 0.100

Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	2.00	4	2.5051E-07	1.0000E-10
2	2.00	4	2.5051E-07	1.0000E-10
3	2.00	4	2.5051E-07	1.0000E-10
4	2.00	4	2.5051E-07	1.0000E-10
5	2.00	4	2.5051E-07	1.0000E-10

Results:

Degree of consolidation	Up [%]	= 34.64
Degree of consolidation	Us [%]	= 34.64
Settlement	s [cm]	= 0.42

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	0.00	0.00
1	0.50	13.50	3.43
2	1.00	27.00	6.79
3	1.50	40.50	9.98
4	2.00	54.00	12.96
5	2.50	50.75	15.65
6	3.00	47.50	18.03
7	3.50	44.25	20.05
8	4.00	41.00	21.72
9	4.50	38.00	23.04
10	5.00	35.00	24.02
11	5.50	32.00	24.69
12	6.00	29.00	25.11
13	6.50	26.50	25.31
14	7.00	24.00	25.35
15	7.50	21.50	25.27

16	8.00	19.00	25.13
17	8.50	18.00	24.98
18	9.00	17.00	24.84
19	9.50	16.00	24.74
20	10.00	15.00	24.71

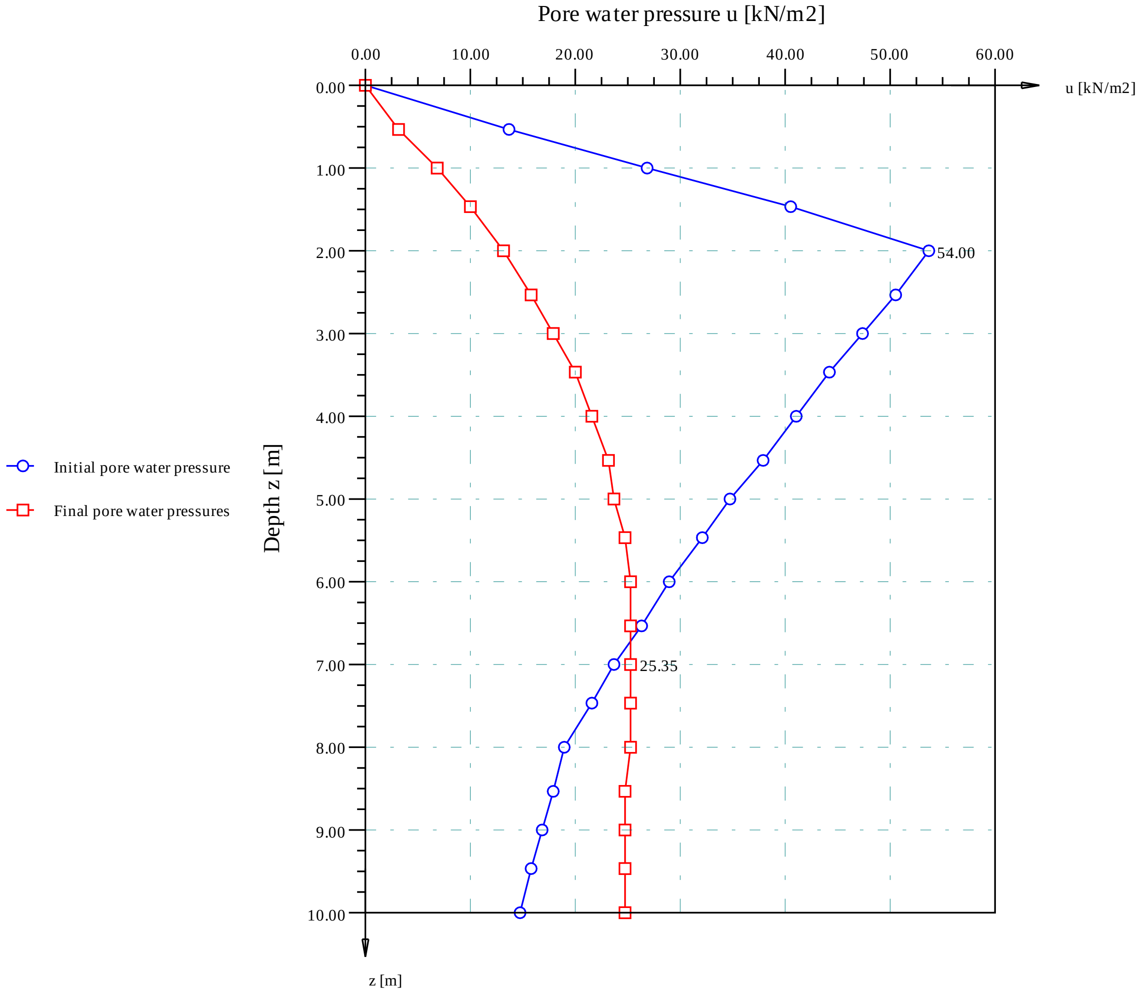
Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
1	0.00	0.00	0.00
2	2.00	54.00	12.96
3	4.00	41.00	21.72
4	6.00	29.00	25.11
5	8.00	19.00	25.13
6	10.00	15.00	24.71

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.50	13.50	11.55	9.09	7.48	6.37	5.56	4.93	4.44	4.04	3.71	3.43
1.00	27.00	22.17	17.57	14.56	12.45	10.89	9.69	8.74	7.96	7.32	6.79
1.50	40.50	31.01	24.90	20.85	17.97	15.80	14.12	12.77	11.67	10.75	9.98
2.00	54.00	37.45	30.70	26.09	22.71	20.12	18.07	16.42	15.06	13.92	12.96
2.50	50.75	41.26	34.80	30.11	26.53	23.71	21.45	19.60	18.05	16.76	15.65
3.00	47.50	42.64	37.20	32.85	29.35	26.52	24.18	22.24	20.61	19.22	18.03
3.50	44.25	42.12	38.07	34.36	31.19	28.52	26.25	24.33	22.69	21.28	20.05
4.00	41.00	40.34	37.69	34.80	32.12	29.74	27.66	25.86	24.29	22.92	21.72
4.50	38.00	37.87	36.39	34.34	32.24	30.26	28.46	26.86	25.43	24.16	23.04
5.00	35.00	35.12	34.49	33.22	31.71	30.19	28.73	27.38	26.14	25.03	24.02
5.50	32.00	32.32	32.25	31.63	30.70	29.64	28.55	27.49	26.49	25.56	24.69
6.00	29.00	29.59	29.89	29.79	29.36	28.74	28.03	27.28	26.53	25.80	25.11
6.50	26.50	27.00	27.56	27.85	27.85	27.63	27.27	26.82	26.33	25.82	25.31
7.00	24.00	24.60	25.37	25.95	26.30	26.42	26.38	26.22	25.98	25.68	25.35
7.50	21.50	22.46	23.39	24.20	24.81	25.22	25.46	25.56	25.54	25.44	25.27
8.00	19.00	20.63	21.70	22.68	23.49	24.13	24.59	24.90	25.08	25.15	25.13
8.50	18.00	19.14	20.34	21.43	22.39	23.20	23.84	24.32	24.66	24.87	24.98
9.00	17.00	18.04	19.33	20.51	21.58	22.50	23.26	23.86	24.32	24.64	24.84

```
     9.50  16.00  17.37  18.72  19.95  21.07  22.06  22.90  23.58  24.10  24.49  24.74
    10.00  15.00  17.15  18.51  19.76  20.90  21.91  22.77  23.48  24.03  24.43  24.71
------------------------------------------------------------------------------------------

Degree of consolidation/ Settlement:
------------------------------------------------------------------------------------------

T [Years]  0.000  0.100  0.200  0.300  0.400  0.500  0.600  0.700  0.800  0.900  1.000

------------------------------------------------------------------------------------------
  Us  [%]   0.00   6.77  12.18  16.52  20.16  23.29  26.04  28.51  30.73  32.77  34.64
   s [cm]   0.00   0.08   0.15   0.20   0.25   0.29   0.32   0.35   0.38   0.40   0.42
------------------------------------------------------------------------------------------
```



Pore water pressure

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 528	**Project: Craig (2007), Example 7.6, page 209**
File: Craig (2007)	**Date: 13-09-2017**
Page No.:	**Title: Consolidation of a single soil layer (SD)**

5.5.3 Example 2: Consolidation of a Single Soil Layer by *EVM*

5.5.3.1 Description of the problem

The determination of eigenvalues and eigenvectors is illustrated by a hand calculation for a very simple example. Consider a clay layer of thickness 10 [m] and coefficient of consolidation C_v = 1.468×10^{-7} [m^2/s] with impervious boundary at the bottom, Figure 5.39. The initial excess pore water pressure on the clay layer is assumed to be uniform u_o=100 [kN/m^2]. It is required to determine the excess pore water pressure after 5 years.

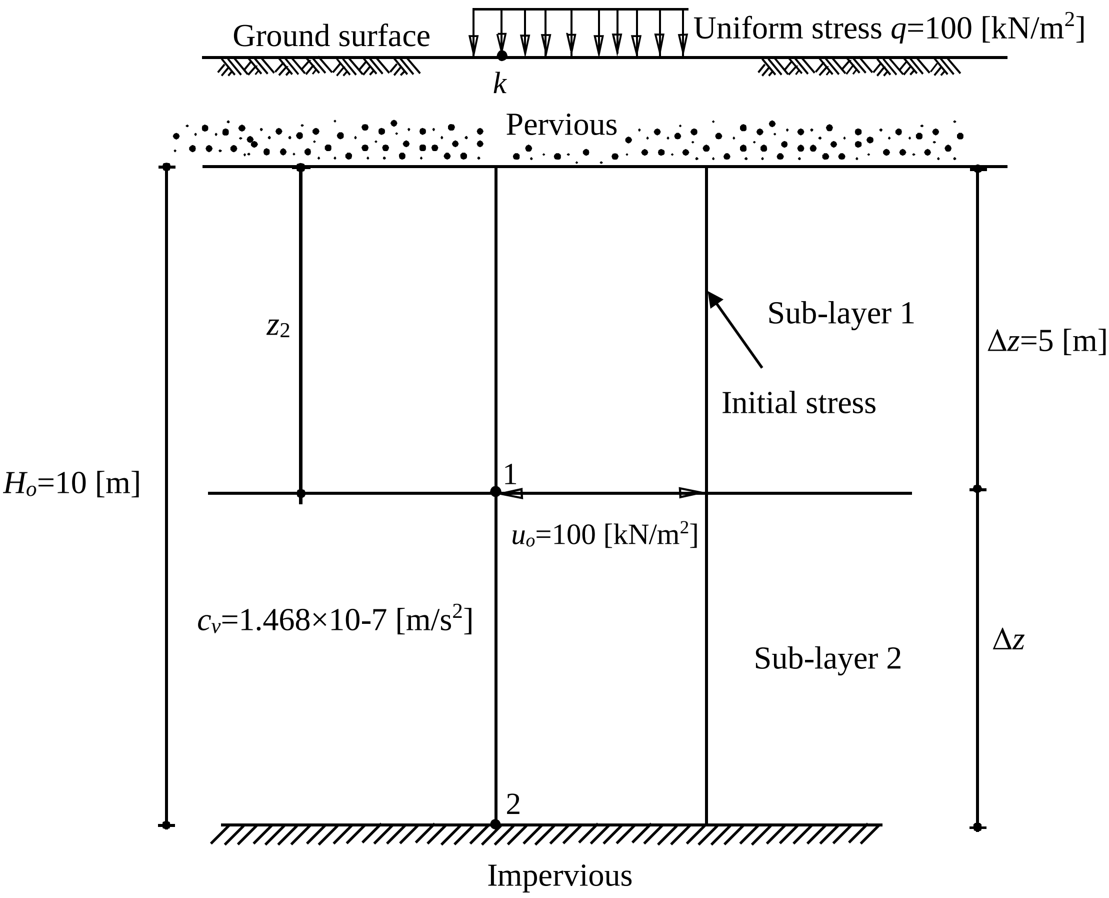


Figure 5.39 Single layer with two sub-layers

5.5.3.2 Analysis of the problem

The whole clay layer is subdivided into two sub-layers, each of 5 [m] with *m* =2 grid nodes as shown in Figure 5.39.

Assume an operator α = 0.02. Then, the time intervals ω is obtained from:

$$\omega = \frac{m^2 c_v T_c}{\alpha H_d^2} = \frac{2^2 \times \left(1.468 \times 10^{-7} \times 31536000\right) \times 5}{0.02 \times 10^2} = 46.295$$

Finite difference equation for a clay layer with an impermeable boundary at the bottom in this case becomes:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = u_o \begin{bmatrix} 1-2\alpha & \alpha \\ 2\alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The characteristic polynomial of [*H*] is given by:

$$p(\lambda) = \det\left[[H] - \lambda[I]\right] = \det \begin{bmatrix} 1-2\alpha-\lambda & \alpha \\ 2\alpha & 1-2\alpha-\lambda \end{bmatrix}$$
$$= (1-2\alpha-\lambda)^2 - 2\alpha^2$$

or

$$(1-2\alpha-\lambda)^2 = 2\alpha^2$$
$$1-2\alpha-\lambda = \pm\sqrt{2}\alpha$$
$$\lambda_{1,2} = 1 - \left(2 \mp \sqrt{2}\right)\alpha$$

For an operator α = 0.02, the eigenvalues are:

$$\lambda_1 = 1 - 0.02\left(2 - \sqrt{2}\right) = 0.9883$$
$$\lambda_2 = 1 - 0.02\left(2 + \sqrt{2}\right) = 0.9317$$

Thus, λ_1=0.9883 and λ_2=0.9317 are the eigenvalues of [*H*].

and the operator matrix [*H*] for an operator α = 0.02 becomes:

$$[H] = \begin{bmatrix} 1-2\alpha & \alpha \\ 2\alpha & 1-2\alpha \end{bmatrix} = \begin{bmatrix} 0.96 & 0.02 \\ 0.04 & 0.96 \end{bmatrix}$$

The eigenvectors $\{\varphi\}_1$ and $\{\varphi\}_2$ corresponding to an eigenvalue λ are obtained by solving the system of linear equations given by:

$$\left[[H] - \lambda[I]\right]\{\varphi\} = 0$$

Computing the eigenvectors corresponding to λ_1=0.9883.

Let $\{\varphi\}_1 = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}$. Then $[[H]-\lambda[I]]\{\varphi\}=0$ gives:

$$\left[\begin{bmatrix} 0.96 & 0.02 \\ 0.04 & 0.96 \end{bmatrix} - 0.9883\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which, the following duplicate equations are obtained:

$$\varphi_1 - 0.707\varphi_2 = 0$$
$$\varphi_1 - 0.707\varphi_2 = 0$$

Let $\varphi_2=t$, then $\varphi_1=0.707t$. Accordingly, eigenvectors corresponding to $\lambda_1=0.9883$ are $\begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix}$

Repeating this process with $\lambda_2=0.9317$, gives:

$$\varphi_1 + 0.707\varphi_2 = 0$$
$$\varphi_1 + 0.707\varphi_2 = 0$$

Let $\varphi_2=t$, then $\varphi_1=-0.707t$. Accordingly, eigenvectors corresponding to $\lambda_1=0.9883$ are $\begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix}$

The corresponding matrix [Φ] and its inverse are:

$$[\Phi] = \begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}$$

$$[\Phi]^{-1} = \begin{bmatrix} 0.707 & 0.5 \\ -0.707 & 0.5 \end{bmatrix}$$

Applying the *EVM* on the operator matrix $[H]^{\omega}$, gives the explicit eigenvalue solution for the excess pore water pressure at time intervals ω.

$$\{u\}_{\omega} = [\Phi][\lambda]^{\omega}[\Phi]^{-1}\{u\}_o$$

or

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{\omega} = 100\begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.9883 & 0 \\ 0 & 0.9317 \end{bmatrix}^{46.295}\begin{bmatrix} 0.707 & 0.5 \\ -0.707 & 0.5 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 100\begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (0.9883)^{46.295} & 0 \\ 0 & (0.9317)^{46.295} \end{bmatrix}\begin{Bmatrix} 1.207 \\ -0.207 \end{Bmatrix}$$

$$= 100\begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.58 & 0 \\ 0 & 0.0378 \end{bmatrix}\begin{Bmatrix} 1.207 \\ -0.207 \end{Bmatrix}$$

$$= 100\begin{bmatrix} 0.707 & -0.707 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} 0.7000 \\ -0.0078 \end{Bmatrix}$$

$$= 100\begin{Bmatrix} 0.5004 \\ 0.6922 \end{Bmatrix}$$

Finally,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{\omega} = \begin{Bmatrix} 50.04 \\ 69.22 \end{Bmatrix}$$

5.5.3.3 Excess pore water pressure by GEO Tools

The input data and results of *GEO Tools* for a single soil layer solved by *EVM* are presented on the next pages. Results of excess pore water pressure *u* after 5 years obtained from *GEO Tools* are compared with those by obtained from hand calculation in Table 5.4. By comparison, one can see a good agreement with hand calculation. The minimum depth increment in z-direction is chosen to be *Di*=9 [m], greater than the layer thickness, to let *GEO Tools* takes the sub layer equal to the entire layer.

Table 5.4 Excess pore water pressure *u* [kN/m^2] after 5 [years]

Depth [m]	Excess pore water pressure *u* [kN/m^2]	
	GEO Tools	Hand calculation
5	50.02	50.04
10	69.17	69.22

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a single soil layer (SD)
Date: 06-09-2017
Project: Simple example by EVM
File: Simple example by EVM

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Eigenvalue method (EVM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 10.00
Depth increment in z-direction       Di          [m]     = 9.00

Time:
Time of consolidation                Tr          [Years] = 5.000
Time increment                       dT          [Years] = 5.000

Generation of times:
Start time                           To          [Years] = 0.000
No. of time intervals                Nt          [-]     = 1
Time interval                        Ti          [Years] = 0.000

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1      10.00          2      1.4680E-07      1.0000E-10
-----------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 57.70
Degree of consolidation        Us [%]  = 57.70
Settlement                     s  [cm] = 4.01
```

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	5.00	100.00	50.02
2	10.00	100.00	69.17

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	0.000	5.000
0.00	100.00	0.00
5.00	100.00	50.02
10.00	100.00	69.17

Degree of consolidation/ Settlement:

T [Years]	0.000	5.000
Us [%]	0.00	57.70
s [cm]	0.00	4.01

5.5.4 Example 3: Consolidation of a Single Soil Layer by *LEM*

5.5.4.1 Description of the problem

A closed form solution for 1-D consolidation of a single layer in an infinite sin series is available in the reference *Terzaghi* and *Peck* (1976). To verify *LEM* for time-dependent settlement problems for deriving a closed form solution, the excess pore water pressure and the degree of consolidation for a single layer calculated analytically by the available closed form solution are compared with those obtained by *LEM*.

5.5.4.2 Analysis of the problem

According to *Terzaghi* and *Peck* (1976), the excess pore water pressure $u(z, t)$ at any depth z and time t can be determined from:

$$u(z,t) = 2u_o \sum_{j=1}^{\infty} \frac{1}{M_j} \sin\left(M_j \xi\right) \exp\left(- M_j^2 T_v\right) \tag{5.138}$$

while the degree of consolidation $U(t)$ is determined from:

$$U(t) = 1 - 2\sum_{j=1}^{\infty} \frac{1}{M_j^2} \exp\left(- M_j^2 T_v\right) \tag{5.139}$$

where:

$M_j = \frac{\pi}{2}(2j - 1)$

$T_v = \frac{c_v t}{H_d^2}$ Time factor in which H_d is the length of drainage pass, for double drainage $H_d = H/2$ while for single drainage $H_d = H$

u_o Initial excess pore water pressure which is constant with depth, [kN/m^2].

To apply *LEM*, the single layer is divided into three equal sub-layers, which gives a grid nodes of $N = 3$. Consequently, an equation in three terms for determining the excess pore water pressure $u(z, t)$ at any depth z and time t is obtained:

$$u(z,t) = u_o \sum_{j=1}^{3} C_j \sin\left(M_j \xi\right) \exp\left(- M_j^2 T_v\right) \tag{5.140}$$

Also an equation in three terms for determining the degree of consolidation $U(t)$ is obtained:

$$U(t) = 1 - \sum_{j=1}^{3} \frac{C_j}{M_j} \exp\left(- M_j^2 T_v\right) \tag{5.141}$$

where $C_1 = \left(2 + \sqrt{3}\right)/3$; $C_2 = 1/3$; and $C_3 = \left(2 - \sqrt{3}\right)/3$

The derivation of the above closed from equations for consolidation by *LEM* is described in the following section.

5.5.4.3 Formulation of excess pore water pressure for single layer by LEM

A partial differential equation such as the consolidation equation can be solved and expressed in series of N terms as:

$$u(z,t)=\sum_{j=1}^{N} C_j \varphi_j(z) B_j(t) \tag{5.142}$$

where:
$u(z, t)$ Excess pore water pressure at any depth z and time t
$\varphi_j(z)$ Set of basis functions in the variable z only
$B_j(z)$ Set functions in the variable t only
C_j Coefficients of basis functions
N Number of function terms

The solution depends on choosing a formula represents the excess pore water pressure along the z-axis and satisfies the boundary conditions. Coefficients of basis functions may be obtained by selecting a set of N arbitrarily points and their function values u_i.

Choosing basis functions are:

$$\varphi_j(z)=\sin(\lambda_j \xi) \tag{5. 143}$$

while functions in the variable t are:

$$\psi_j(z)=\exp(\lambda_j T_v) \tag{5.144}$$

where:
ξ Depth ratio, $\xi=\dfrac{z}{H}$ with $0 \le \xi \le 1$
z Vertical coordinate, [m]
H Layer thickness, [m].
T_v Time factor, $T_v=\dfrac{c_v t}{H^2}$
t Consolidation time, [year]
c_v Coefficient of consolidation, [m^2/year]
λ_i Deferential equation operators

Equation (5.142) is rewritten in matrix form as:

$$\{u\}=[\varphi]\{\psi\} \tag{5.145}$$

where $[\varphi]=[\varphi_1(z) \quad \varphi_2(z) \quad \dots \quad \varphi_N(z)]$ and $\{\psi\}^T=\{C_1\psi_1(t) \quad C_2\psi_2(t) \quad \dots \quad C_N\psi_N(t)\}$

To formulate the analysis, assume the clay layer of thickness H_o shown in Figure 5.40. The layer is divided into 3 equal intervals of thickness Δz.

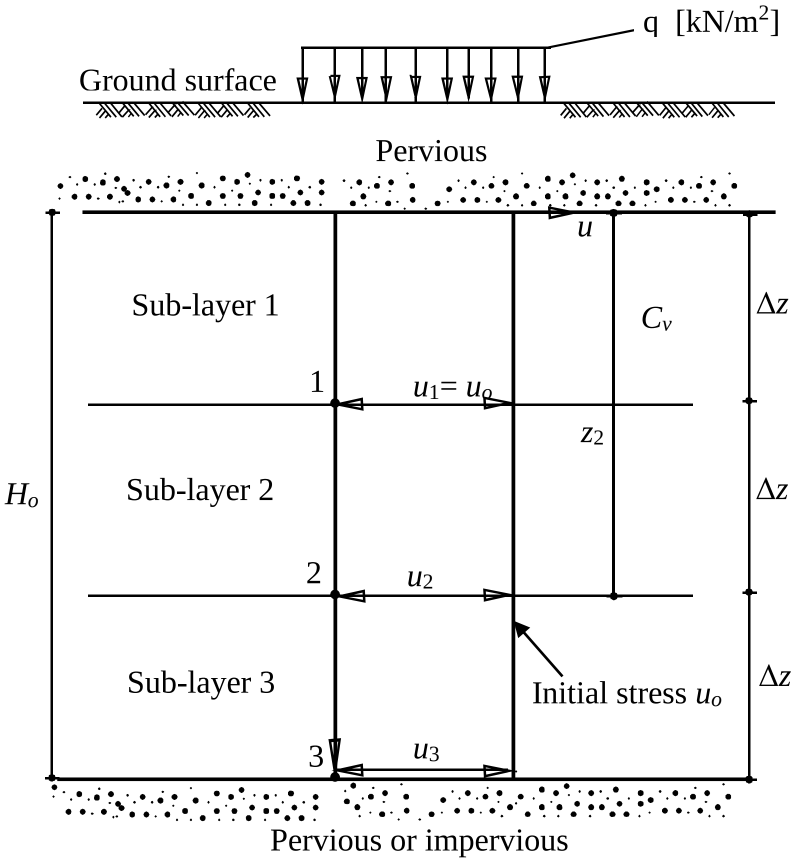


Figure 5.40 Single layer with three sub-layers

Consider the clay layer with free drainage at the top and impervious bottom boundary. For previous top boundary the excess pore water pressure at the top boundary is known and equal zero. In order to simplify the notations and without lose of generality, a sufficient grid points of N = 3 in the layer is considered.

Let the excess pore water pressure u at any time along the clay layer depth can be expressed by the following formula of variables depth z and time t:

$$u = C_1 \sin(M_1 \xi) \exp(-M_1^2 T_v) + C_2 \sin(M_2 \xi) \exp(-M_2^2 T_v) + C_3 \sin(M_3 \xi) \exp(-M_3^2 T_v) \tag{5.146}$$

where:

ξ Depth ratio, $\xi = z/H_d$ with $0 \le \xi \le 1$

z Vertical coordinate, [m]

H_d Length of drainage, [m].
For double drainage $H_d = H_o/2$ while for single drainage $H_d = H_o$

T_v Time factor, $T_v = \dfrac{c_v t}{H_d^2}$

t Consolidation time, [year]

C_v Coefficient of consolidation, [m^2/year]

Choosing $M_j = \frac{\pi}{2}(2j\text{-}1)$, lets the above equation represents the partial differential equation of consolidation $\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$ and satisfies the boundary condition at the top u_{top} = 0 and at the axis of symmetry for double drainage and at the bottom for single drainage $\frac{\partial u}{\partial z} = 0$.

Equation coefficients C_1 to C_3 can be found using the initial condition. Consider the initial excess pore water pressure at the beginning of consolidation at time t = 0 is uniform on the clay layer and equal to u_o. At time t = 0, Eq. (5.146) becomes:

$$u_o = C_1 \sin(M_1\xi) + C_2 \sin(M_2\xi) + C_3 \sin(M_3\xi) \tag{5.147}$$

Consider a system of linear equations at a set of grid points of N = 3 as follows:

$$\begin{Bmatrix} u_o \\ u_o \\ u_o \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \\ 2 & -2 & 2 \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{5.148}$$

Through inverting the matrix:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \frac{u_o}{3}\begin{Bmatrix} 2+\sqrt{3} \\ 1 \\ 2-\sqrt{3} \end{Bmatrix} \tag{5.149}$$

Substituting these coefficients into Eq. (5.146), gives the excess pore water pressure. Equation (5.146) gives the same analytical solution of one dimensional consolidation after *Terzaghi* (1976). However this equation is derived from N grid points, it can give directly the excess pore water pressure at any depth z in the clay layer with any time t. Eq. (5.146) can be rewritten in general form as:

$$u(z,t) = u_o \sum_{j=1}^{3} C_j \sin(M_j\xi)\ \exp(\text{-}\ M_j^2 T_v) \tag{5.150}$$

5.5.4.4 Degree of consolidation

Integrating Eq. (5.150) over the entire clay layer, gives the average excess pore water pressure in the layer at the required time as:

$$\Delta u = \frac{1}{H_o}\int_0^{H_o} u\, dz = \frac{2u_o}{3\pi}\left[\left(2+\sqrt{3}\right)\exp\left(-M_1^2 T_v\right) + \exp\left(-M_2^2 T_v\right) + \left(2-\sqrt{3}\right)\exp\left(-M_3^2 T_v\right)\right] \tag{5.151}$$

but, the degree of consolidation is expressed as:

$$U = 1 - \frac{\Delta u}{u_o} \tag{5.152}$$

where:

U	Degree of consolidation at time t, [-]
Δu	Average excess pore water pressure in the entire clay layer at time t, [kN/m^2]

Substituting Eq. (5.151) into Eq. (5.152), gives the degree of consolidation as:

$$U = 1 - \frac{2}{3\pi}\left[\left(2+\sqrt{3}\right)\exp\left(-M_1^2 T_v\right) + \exp\left(-M_2^2 T_v\right) + \left(2-\sqrt{3}\right)\exp\left(-M_3^2 T_v\right)\right] \tag{5.153}$$

5.5.4.5 Results:

Results of *LEM* are compared with those of the closed form solution. Figure 5.41 to Figure 5.43 show excess pore water pressure ratios and degrees of consolidation at different time factors. However, the above two equations of *LEM* are derived in three terms; results obtained from *LEM* are in a good agreement with those of the analytical solution of *Terzaghi* for both cases of double and single drainage.

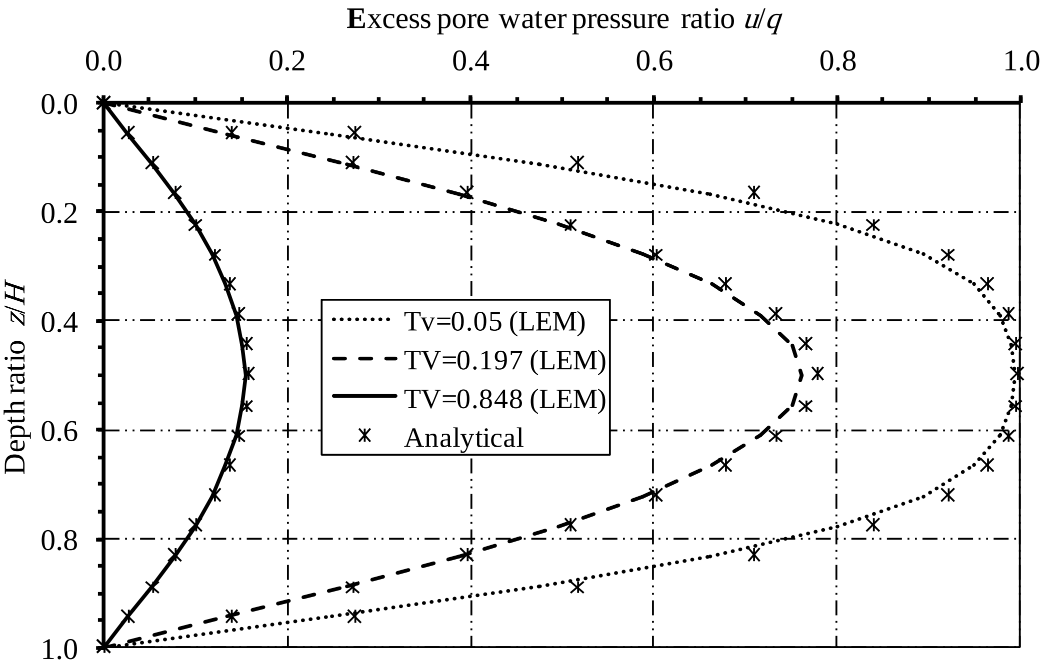


Figure 5.41 Excess pore water pressure ratio with depth ratio at different T_v for double drainage layer

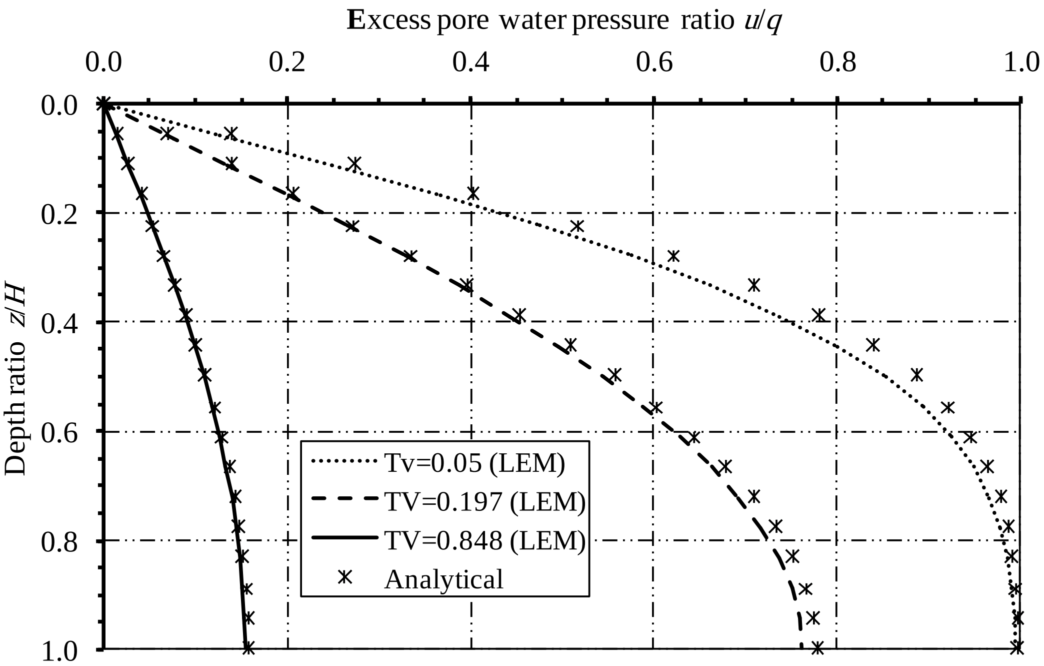


Figure 5.42 Excess pore water pressure ratio with depth ratio at different T_v for single drainage layer

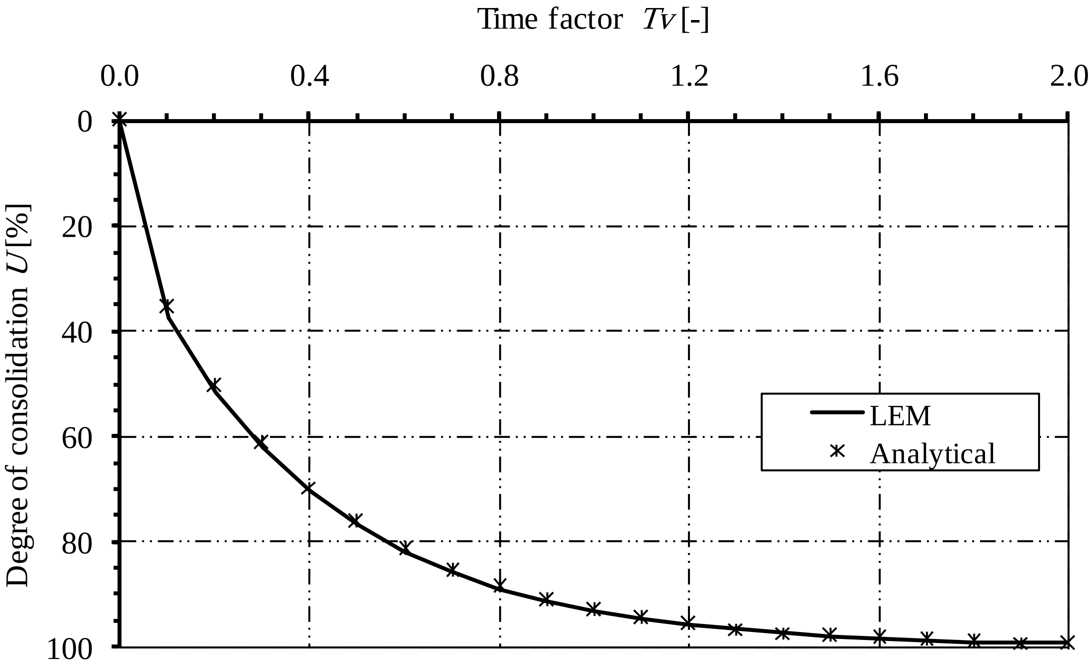


Figure 5.43 Degree of consolidation at different T_v

5.5.4.6 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for single and double soil layers are presented on the next pages.

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a single soil layer (DD)
Date: 06-09-2017
Project: Deriving closed form solution for a single layer
File: Single layer (DD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------
Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo         [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 18.00
Depth increment in z-direction       Di         [m]     = 1.00

Time:
Time of consolidation                Tr         [Years] = 1.100

Generation of times:
Start time                           To         [Years] = 0.300

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              0.790
  2              3.520
----------------------

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1      18.00         18      4.7560E-07      5.0000E-10
-----------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 50.99
Degree of consolidation        Us [%]  = 50.99
Settlement                     s  [cm] = 9.84
```

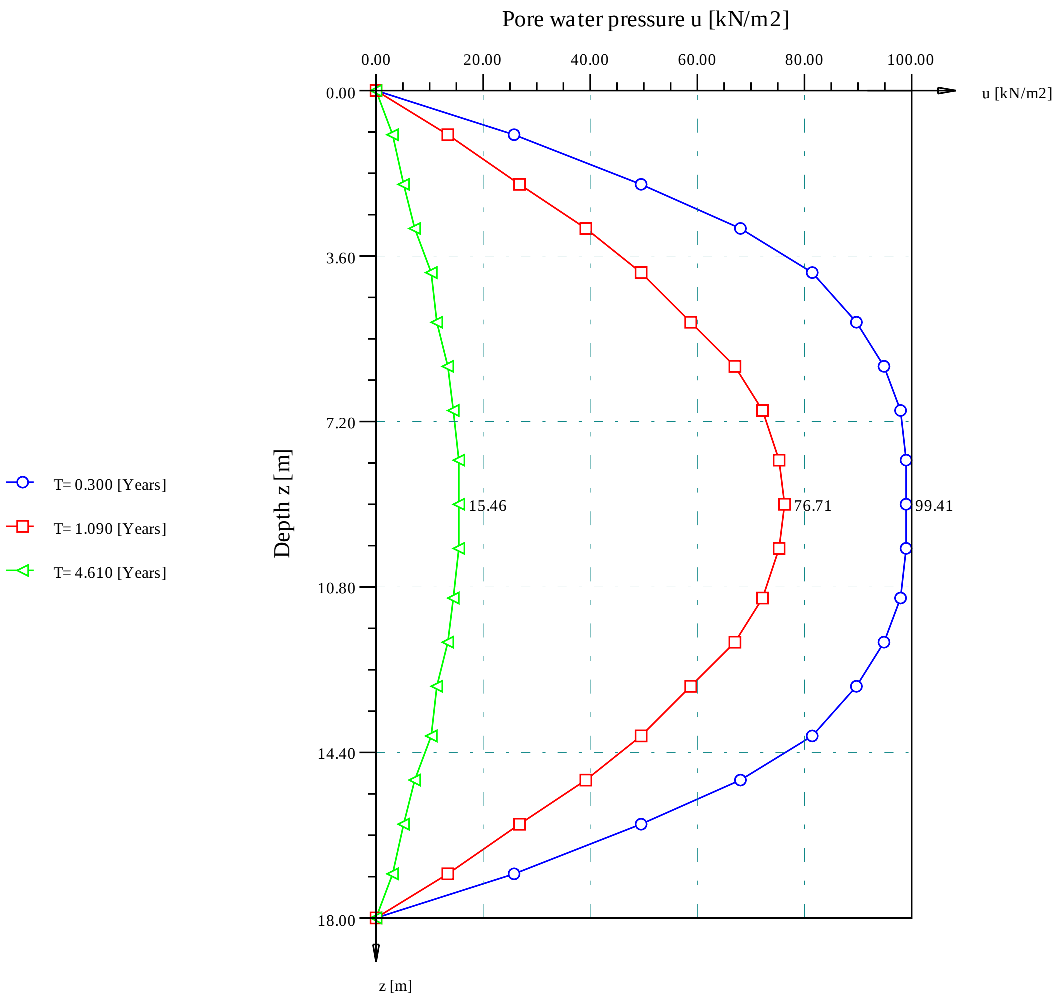
Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	1.00	100.00	13.57
2	2.00	100.00	26.67
3	3.00	100.00	38.87
4	4.00	100.00	49.78
5	5.00	100.00	59.08
6	6.00	100.00	66.54
7	7.00	100.00	71.97
8	8.00	100.00	75.28
9	9.00	100.00	76.38
10	10.00	100.00	75.28
11	11.00	100.00	71.97
12	12.00	100.00	66.54
13	13.00	100.00	59.08
14	14.00	100.00	49.78
15	15.00	100.00	38.87
16	16.00	100.00	26.67
17	17.00	100.00	13.57
18	18.00	100.00	0.00

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	0.300	1.090	4.610
0.00	0.00	0.00	0.00
1.00	25.88	13.64	2.68
2.00	49.11	26.80	5.29
3.00	67.82	39.06	7.73
4.00	81.35	50.02	9.93
5.00	90.13	59.36	11.84
6.00	95.24	66.84	13.38
7.00	97.90	72.29	14.52
8.00	99.08	75.61	15.22
9.00	99.41	76.71	15.46
10.00	99.08	75.61	15.22
11.00	97.90	72.29	14.52
12.00	95.24	66.84	13.38
13.00	90.13	59.36	11.84
14.00	81.35	50.02	9.93
15.00	67.82	39.06	7.73
16.00	49.11	26.80	5.29
17.00	25.88	13.64	2.68
18.00	0.00	0.00	0.00

```
Degree of consolidation/ Settlement:
------------------------------------------------

T [Years]      0.300      1.090      4.610

------------------------------------------------
  Us  [%]      26.84      50.76      90.16
   s [cm]       5.18       9.79      17.39
------------------------------------------------
```



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1033 **File: Single layer (DD)** **Page No.:**	**Project: Deriving closed form solution for a single layer** **Date: 06-09-2017** **Title: Consolidation of a single soil layer (DD)**

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a single soil layer (SD)
Date: 06-09-2017
Project: Deriving closed form solution for a single layer
File: Single layer (SD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------
Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure        uo          [kN/m2] = 100.00
Overburden pressure                 Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                     Hb          [m]     = 18.00
Depth increment in z-direction      Di          [m]     = 1.00

Time:
Time of consolidation               Tr          [Years] = 4.330

Generation of times:
Start time                          To          [Years] = 1.100

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              3.230
  2             14.330
----------------------

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1      18.00         18      4.7560E-07      5.0000E-10
-----------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 50.50
Degree of consolidation        Us [%]  = 50.50
```

Settlement s [cm] = 9.74

Initial and Final pore water pressures with depth:

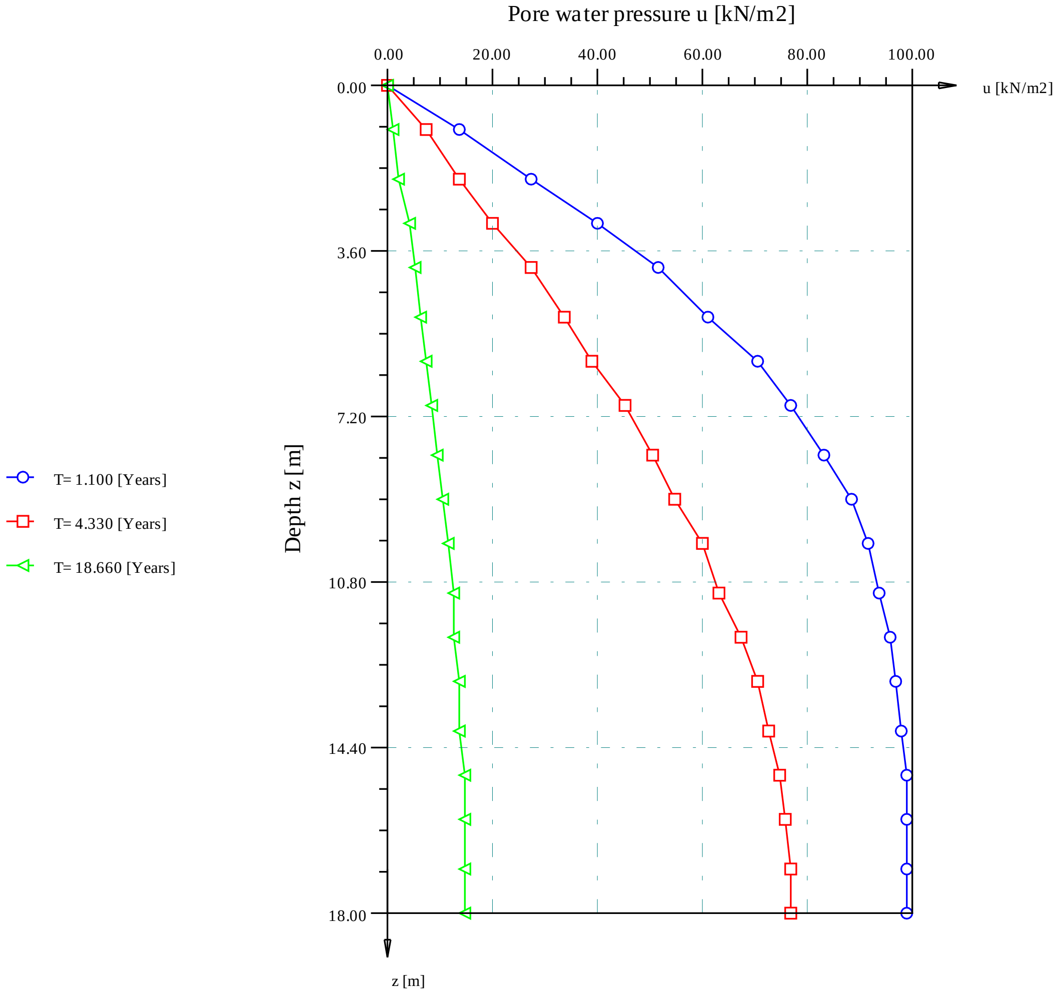
No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	1.00	100.00	6.89
2	2.00	100.00	13.72
3	3.00	100.00	20.43
4	4.00	100.00	26.97
5	5.00	100.00	33.27
6	6.00	100.00	39.29
7	7.00	100.00	44.98
8	8.00	100.00	50.30
9	9.00	100.00	55.22
10	10.00	100.00	59.69
11	11.00	100.00	63.69
12	12.00	100.00	67.20
13	13.00	100.00	70.20
14	14.00	100.00	72.67
15	15.00	100.00	74.60
16	16.00	100.00	75.99
17	17.00	100.00	76.83
18	18.00	100.00	77.10

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	1.100	4.330	18.660
0.00	0.00	0.00	0.00
1.00	13.79	6.89	1.32
2.00	27.16	13.72	2.62
3.00	39.76	20.43	3.91
4.00	51.27	26.97	5.16
5.00	61.47	33.27	6.38
6.00	70.25	39.29	7.55
7.00	77.58	44.98	8.66
8.00	83.52	50.30	9.71
9.00	88.19	55.22	10.68
10.00	91.75	59.69	11.57
11.00	94.39	63.69	12.37
12.00	96.28	67.20	13.08
13.00	97.60	70.20	13.69
14.00	98.48	72.67	14.19
15.00	99.05	74.60	14.59
16.00	99.40	75.99	14.87
17.00	99.59	76.83	15.04
18.00	99.65	77.10	15.10

Degree of consolidation/ Settlement:

T [Years]	1.100	4.330	18.660
Us [%]	25.53	50.50	90.39
s [cm]	4.92	9.74	17.44



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1056 File: Single layer (SD) Page No.:	Project: Deriving closed form solution for a single layer Date: 06-09-2017 Title: Consolidation of a single soil layer (SD)

5.5.5 Example 4: Consolidation of a Double-Layered Soil

5.5.5.1 Description of the problem:

To illustrate the hand application of *LEM,* the excess pore water pressure for a double-layered soil shown in 0 after one year is determined and compared with that obtained by *FDM*. The two layers are equal in thickness, each of $h_1=h_2=9$ [m]. The coefficient of consolidation for the first layer is $c_{v1}=100$ [m^2/Year], while that for the second layer is $c_{v2}=25$ [m^2/Year]. The coefficient of permeability ratio for the two layers is $\eta=k_2/k_1=0.25$. The initial excess pore water pressure is distributed uniformly on the soil layers and equal to $u_o=100$ [kN/m^2]. Pervious top and bottom boundaries are assumed.

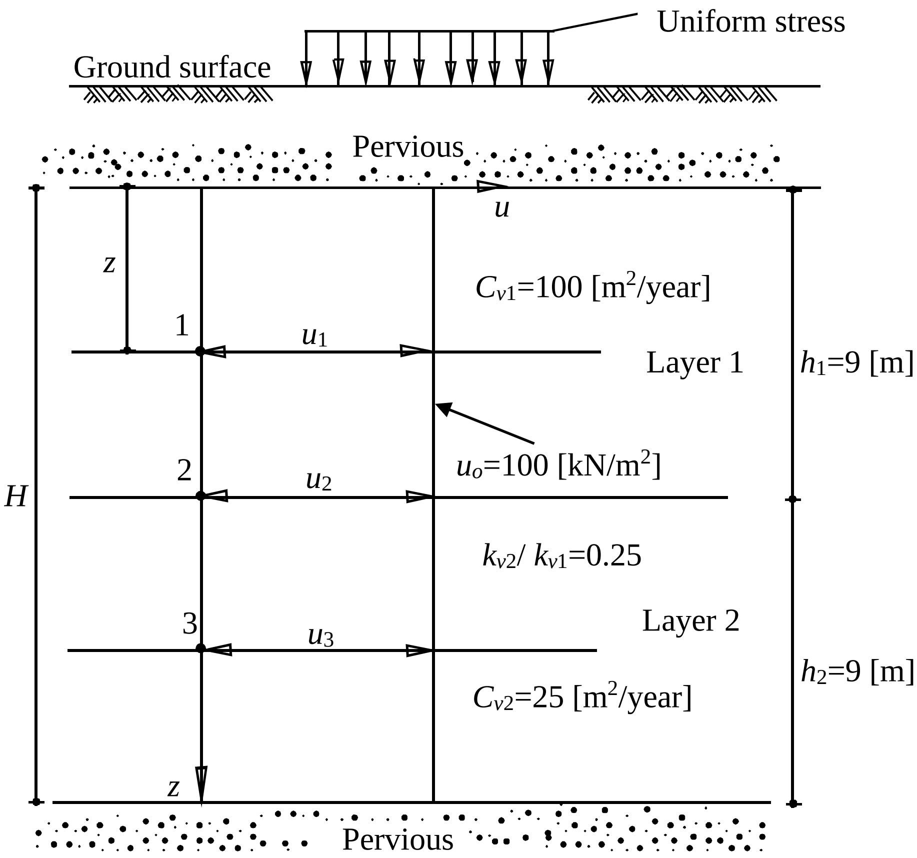


Figure 5.44 Two clay layers with soil properties

5.5.5.2 Analysis of the problem:

Equations of excess pore water pressure for the two layers can be expressed as:

$$u_1(z,t) = \sum_{j=1}^{N=3} C_j \left[A_{1j} \sin(\lambda_j \xi) + B_{1j} \cos(\lambda_j \xi)\right] \exp(-\lambda_j^2 T_{v1})$$
$$u_2(z,t) = \sum_{j=1}^{N=3} C_j \left[A_{2j} \sin(\mu\lambda_j \rho) + B_{2j} \cos(\mu\lambda_j \rho)\right] \exp(-\mu^2 \lambda_j^2 T_{v2}) \quad (5.154)$$

where:

$u_i(z, t)$	Excess pore water pressure at any depth z and time t for layer i, [kN/m^2];
C_j	Constants of basis functions;
A_{ij} and B_{ij}	Coefficients of basis functions for layer i;
λ_j	Differential equation operator;
c_{vi}	Coefficient of consolidation of layer i, [Year/m^2];
t	Time for which excess pore water pressure is computed, [Year];
z	Vertical coordinate, [m];
h_i	Thickness of layer i, [m];
ξ	Depth ratio in local coordinate of layer 1, $\xi = \frac{z}{h_1}$ with $0 \le \xi \le 1$;
ρ	Depth ratio in local coordinate of layer 2, $\rho = \frac{z}{h_2}$ with $0 \le \rho \le 1$;
T_{vi}	Time factor for layer i, $T_{vi} = \frac{c_{vi} t}{h_i^2}$.

Satisfying free drainage at the top $u_1(0, t) = 0$, requires that:

$$\begin{Bmatrix} A_{1j} \\ B_{1j} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad (5.155)$$

Equation (5.154) for the first layer becomes:

$$u_1(z,t) = \sum_{j=1}^{N=3} C_j \sin(\lambda_j \xi) \exp(-\lambda_j^2 T_{v1}) \quad (5.156)$$

or

$$u_1(z,t) = C_1 \sin(\lambda_1 \xi) \exp(-\lambda_1^2 T_{v1}) + C_2 \sin(\lambda_2 \xi) \exp(-\lambda_2^2 T_{v1}) + C_3 \sin(\lambda_3 \xi) \exp(-\lambda_3^2 T_{v1}) \quad (5.157)$$

To satisfy the solution condition, the following equations should be satisfied:

$$\lambda_j^2 T_{v1} = \mu^2 \lambda_j^2 T_{v2} \quad (5.158)$$

$$\lambda_j^2 \frac{c_{v1} t}{h_1^2} = \mu^2 \lambda_j^2 \frac{c_{v2} t}{h_2^2} \quad (5.159)$$

$$\mu = \frac{h_2}{h_1}\sqrt{\frac{c_{v1}}{c_{v2}}} = \frac{9}{9}\sqrt{\frac{100}{25}} = 2 \tag{5.160}$$

Equation (5.154) for the second layer becomes:

$$\begin{aligned} u_2(z,t) &= C_1\left[A_{21}\sin(\mu\lambda_1\rho) + B_{21}\cos(\mu\lambda_1\rho)\right]\ \exp(-\mu^2\lambda_1^2 T_{v2}) \\ &+ C_2\left[A_{22}\sin(\mu\lambda_2\rho) + B_{22}\cos(\mu\lambda_2\rho)\right]\ \exp(-\mu^2\lambda_2^2 T_{v2}) + \\ &C_3\left[A_{23}\sin(\mu\lambda_2\rho) + B_{23}\cos(\mu\lambda_2\rho)\right]\ \exp(-\mu^2\lambda_2^2 T_{v2}) \end{aligned} \tag{5.161}$$

5.5.5.3 Determining coefficients A_{2j} and B_{2j}:

Depth ratio in local coordinate at the top of the layer *i* is given by:

$$\xi_t = \frac{z_t}{h_i} = \frac{0}{h_i} = 0, \rho_t = 0 \tag{5.162}$$

while that at the bottom of the layer is given by:

$$\xi_b = \frac{z_b}{h_i} = \frac{h_i}{h_i} = 1, \rho_b = 1 \tag{5.163}$$

For simplicity when formulating excess pore water equations, the following parameters are defined:

$$\theta = \lambda_j \xi_b,\ \beta = \mu\lambda_j \rho_t \tag{5.164}$$

Substituting Eqns. (5.162) and (5.163) into Eq. (5.164) gives:

$$\theta = \lambda_j,\ \beta = 0 \tag{5.165}$$

Relations among coefficients A_{2j} and B_{2j} can be obtained using interface and boundary conditions. Equating the excess pore water pressures $u_1(z, t) = u_2(z, t)$ at layer interfaces, leads to:

$$\sin(\theta) = A_{2j}\sin(\beta) + B_{2j}\cos(\beta) \tag{5.166}$$

while equating the vertical velocity of flow $k_{v1}\left(\frac{\partial u}{\partial z}\right)_1 = k_{v2}\left(\frac{\partial u}{\partial z}\right)_2$ at layer interfaces, leads to:

$$\eta\cos(\theta) = A_{2j}\cos(\beta) - B_{2j}\sin(\beta) \tag{5.167}$$

where:

$$\eta = \frac{h_2 k_{v1}}{\mu h_1 k_{v2}} = \frac{k_{v1}}{k_{v2}} \sqrt{\frac{c_{v2}}{c_{v1}}} = \frac{4}{1}\sqrt{\frac{25}{100}} = 2 \tag{5.168}$$

Substituting values of θ_i and β_i into Eqns. (5.166) and (5.167) gives:

$$\sin(\lambda_j) = B_{2j} \tag{5.169}$$

$$\eta \cos(\lambda_j) = A_{2j} \tag{5.170}$$

Apply the boundary condition for double drainage at the bottom $u_2(h_2, t)=0$, leads to:

$$A_{2j} \sin(\mu\lambda_j) + B_{2j} \cos(\mu\lambda_j) = 0 \tag{5.171}$$

From Eqns (5.169) to (5.171), the following characteristic equation in the unknown Eigen values λ_j (differential equation operators) can be obtained:

$$\eta \cos(\lambda_j)\sin(\mu\lambda_j) + \sin(\lambda_j)\cos(\mu\lambda_j) = 0 \tag{5.172}$$

The operator λ_j is the positive root of the above Eigen equation.

$$2\cos(\lambda_j)\sin(2\lambda_j) + \sin(\lambda_j)\cos(2\lambda_j) = 0 \tag{5.173}$$

$$2\cos(\lambda_j)\ [2\sin(\lambda_j)\cos(\lambda_j)]\ + \sin(\lambda_j)\cos(2\lambda_j) = 0 \tag{5.174}$$

$$4\cos^2(\lambda_j) + \cos(2\lambda_j) = 0 \tag{5.175}$$

$$2[1 + \cos(2\lambda_j)]\ + \cos(2\lambda_j) = 0 \tag{5.176}$$

$$\cos(2\lambda_j) = \frac{-2}{3} \tag{5.177}$$

λ_j is the positive roots of the above Eigen equation.

$$\lambda_j = 1.1505,\ 1.991,\ 4.292 \tag{5.178}$$

Substituting the value of λ_j obtained from Eq. (5.178) into Eqns (5.170) and (5.171), gives coefficients A_{2j} and B_{2j}.

$$A_{2j} = \eta\cos(\lambda_j) = 2\cos(\lambda_j)$$
$$A_{2j} = 0.816,\ -0.816,\ -0.816$$
$$B_{2j} = \sin(\lambda_j) \tag{5.179}$$
$$B_{2j} = 0.913,\ 0.913,\ -0.913$$

Constant C_j can be found using the initial condition. Consider a system of linear equations at a set of N =3 grid nodes at time t = 0 as follows:

At node 1

$$u_1(z,t) = C_1\sin(\lambda_1\xi) + C_2\sin(\lambda_2\xi) + C_3\sin(\lambda_3\xi) \tag{5.180}$$

$$u_o = C_1\sin(1.150\times 0.5) + C_2\sin(1.991\times 0.5) + C_3\sin(4.292\times 0.5) \tag{5.181}$$

$$u_o = 0.544C_1 + 0.839C_2 + 0.839C_3 \tag{5.182}$$

At node 2

$$u_o = C_1\sin(1.150\times 1) + C_2\sin(1.991\times 1) + C_3\sin(4.292\times 1) \tag{5.183}$$

$$u_o = 0.913C_1 + 0.913C_2 - 0.913C_3 \tag{5.184}$$

At node 3

$$\begin{aligned} u_2(z,t) = {} & C_1[A_{21}\sin(\mu\lambda_1\rho) + B_{21}\cos(\mu\lambda_1\rho)] \\ & + C_2[A_{22}\sin(\mu\lambda_2\rho) + B_{22}\cos(\mu\lambda_2\rho)] + \\ & C_3[A_{23}\sin(\mu\lambda_2\rho) + B_{23}\cos(\mu\lambda_2\rho)] \end{aligned} \tag{5.185}$$

$$\begin{aligned} u_o = {} & C_1[0.816\sin(2\times 1.150\times 0.5) + 0.913\cos(2\times 1.150\times 0.5)] \\ & + C_2[-0.816\sin(2\times 1.991\times 0.5) + 0.913\cos(2\times 1.991\times 0.5)] + \\ & C_3[-0.816\sin(2\times 4.292\times 0.5) - 0.913\cos(2\times 4.292\times 0.5)] \end{aligned} \tag{5.186}$$

$$u_o = 1.119C_1 - 1.119C_2 + 1.119C_3 \tag{5.187}$$

Eqns (5.182), (5.184) and (5.187) in matrix form are:

$$\begin{Bmatrix} u_o \\ u_o \\ u_o \end{Bmatrix}_o = \begin{bmatrix} 0.544 & 0.839 & 0.839 \\ 0.913 & 0.913 & -0.913 \\ 1.119 & -1.119 & 1.119 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{5.188}$$

Through inverting the matrix:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = u_o \begin{Bmatrix} 0.994 \\ 0.324 \\ 0.224 \end{Bmatrix} \tag{5.189}$$

5.5.5.4 Time factor:

$$T_{v1} = \frac{c_{v1} t}{h_1^2} = \frac{100 \times 1.0}{9 \times 9} = 1.235 \tag{5.190}$$

5.5.5.5 Excess pore water pressure:

Excess pore water pressure for nodes 1, 2 and 3 of the two clay layers can be expressed as:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 100 \begin{bmatrix} 0.544 & 0.839 & 0.839 \\ 0.913 & 0.913 & -0.913 \\ 1.119 & -1.119 & 1.119 \end{bmatrix}$$

$$\begin{bmatrix} \exp\left(-1.150^2 \times 1.235\right) & 0 & 0 \\ 0 & \exp\left(-1.991^2 \times 1.235\right) & 0 \\ 0 & 0 & \exp\left(-4.292^2 \times 1.235\right) \end{bmatrix} \begin{Bmatrix} 0.994 \\ 0.324 \\ 0.224 \end{Bmatrix} \tag{5.191}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 100 \begin{bmatrix} 0.544 & 0.839 & 0.839 \\ 0.913 & 0.913 & -0.913 \\ 1.119 & -1.119 & 1.119 \end{bmatrix} \begin{bmatrix} 0.195 & 0 & 0 \\ 0 & 0.007 & 0 \\ 0 & 0 & 1 \times 10^{-10} \end{bmatrix} \begin{Bmatrix} 0.994 \\ 0.324 \\ 0.224 \end{Bmatrix} \tag{5.192}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 0.544 & 0.839 & 0.839 \\ 0.913 & 0.913 & -0.913 \\ 1.119 & -1.119 & 1.119 \end{bmatrix} \begin{Bmatrix} 19.383 \\ 0.227 \\ -2 \times 10^{-9} \end{Bmatrix} \tag{5.193}$$

The values of excess pore water pressure at the three nodes using *LEM* obtained from Eq. (5.193) are:

$$\{u_1 \quad u_2 \quad u_3\}^T = \{10.73 \quad 17.90 \quad 21.44\}^T \tag{5.194}$$

The same example is analyzed using the *FDM* with 0.9 [m] depth increment. Results obtained from *FDM* after one year are:

$$\{u_1 \quad u_2 \quad u_3\}^T = \{10.23 \quad 16.89 \quad 22.46\}^T \tag{5.195}$$

Furthermore, the excess pore water pressure with depth at different times obtained from *LEM* and *FDM* are compared in Figure 5.45. The comparison shows a good agreement between the two results.

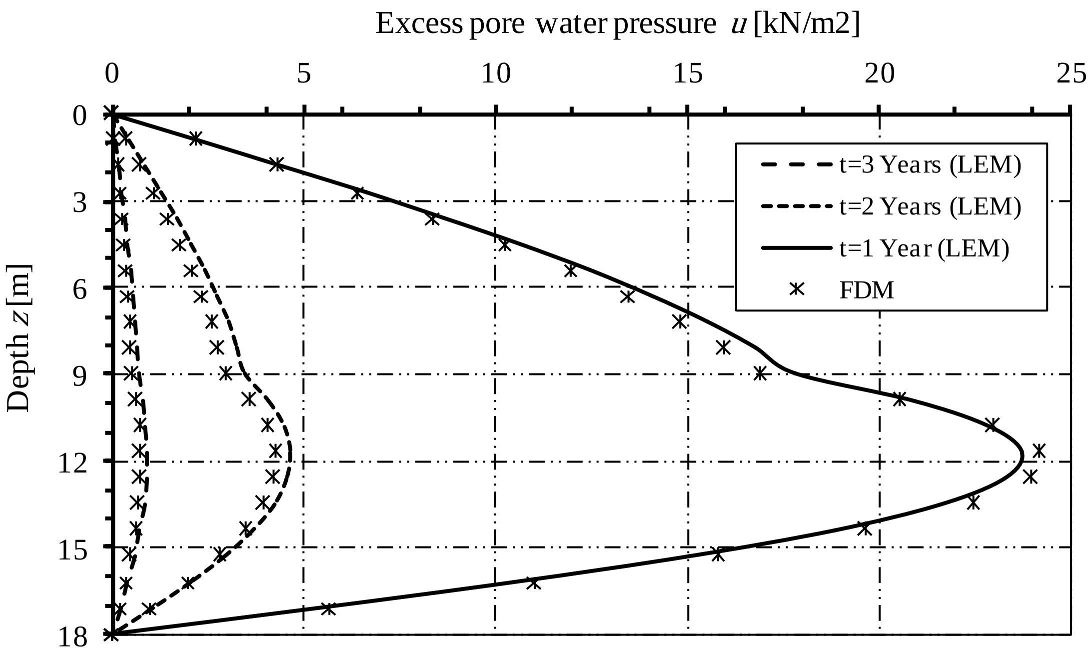


Figure 5.45 Excess pore water pressure with depth at different times

5.5.5.6 Examination of the total number of sub-layers:

The same example is tested for different total numbers of sub layers. Comparison between results of the degree of consolidation *U* calculated by *LEM* and *FDM*, shows that an accurate value of *U* can be obtained after a small number of sub layers by *LEM*, Figure 5.46.

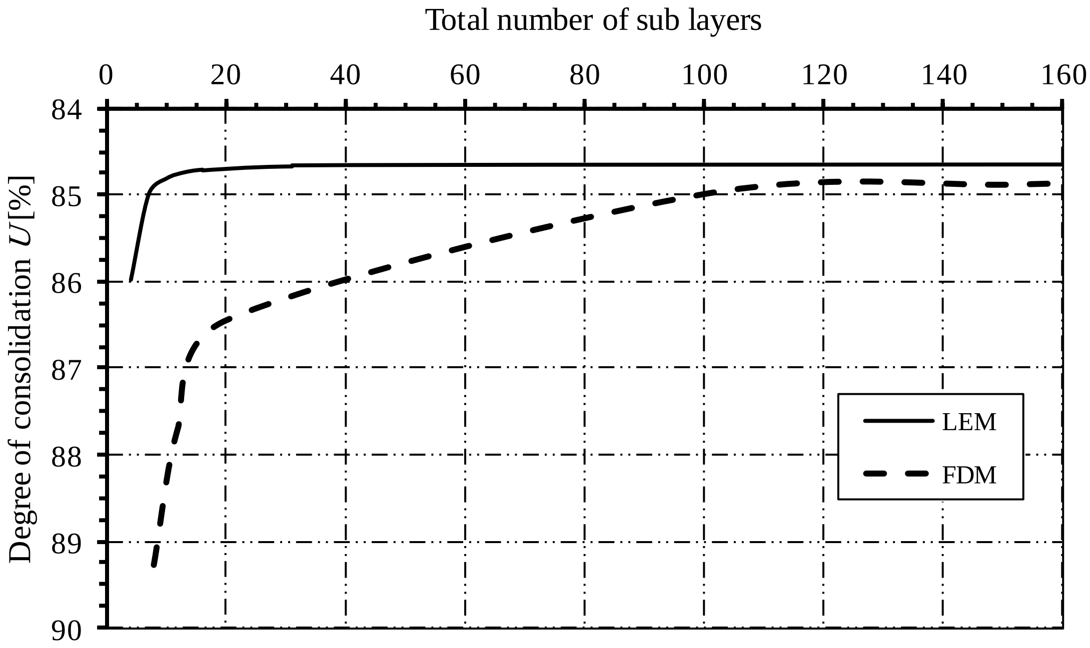


Figure 5.46 Degree of consolidation *U* [%] with total number of sub-layers

5.5.5.7 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation by *LEM* and *FDM* are presented on the next pages. For *LEM*, it is sufficient to chose the depth increment in *z*-direction equal to the thickness of the layer 9 [m], while for *FDM*, the depth increment in z-direction is chosen to be 0.9 [m]. By comparison, one can see a good agreement with hand calculation.

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Double-layered soil (1 Sublayer)
Date: 06-09-2017
Project: To illustrate the hand application of LEM (Di = 9 [m])
File: 2 Layers-Di=9m

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 18.00
Depth increment in z-direction       Di          [m]     = 9.00

Time:
Time of consolidation                Tr          [Years] = 1.000
```

Generation of times:

Start time	To	[Years]	= 1.000
No. of time intervals	Nt	[-]	= 2
Time interval	Ti	[Years]	= 1.000

Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	9.00	1	3.1710E-06	4.0000E-09
2	9.00	3	7.9270E-07	1.0000E-09

Results:

Degree of consolidation	Up [%]	= 85.98
Degree of consolidation	Us [%]	= 85.98
Settlement	s [cm]	= 19.90

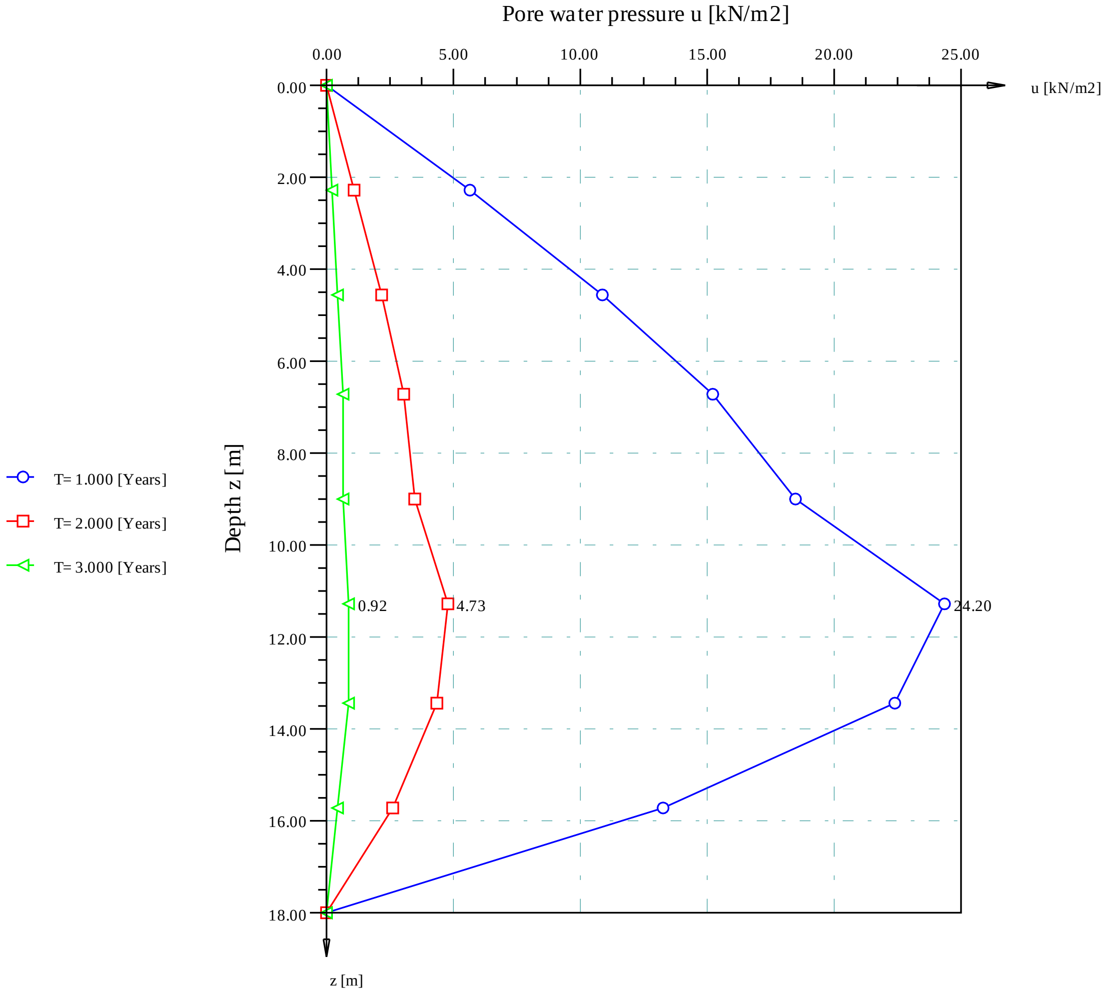
Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	2.25	100.00	5.70
2	4.50	100.00	10.93
3	6.75	100.00	15.25
4	9.00	100.00	18.31
5	11.25	100.00	24.20
6	13.50	100.00	22.30
7	15.75	100.00	13.27
8	18.00	100.00	0.00

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	1.000	2.000	3.000
0.00	0.00	0.00	0.00
2.25	5.70	1.11	0.22
4.50	10.93	2.12	0.41
6.75	15.25	2.97	0.58
9.00	18.31	3.57	0.70
11.25	24.20	4.73	0.92

```
    13.50      22.30       4.37       0.85
    15.75      13.27       2.60       0.51
    18.00       0.00       0.00       0.00
----------------------------------------------

Degree of consolidation/ Settlement:
----------------------------------------------

T [Years]      1.000      2.000      3.000

----------------------------------------------
  Us  [%]      85.98      97.26      99.47
   s [cm]      19.90      22.51      23.02
----------------------------------------------
```



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 **PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323**	
Scale: 216	**Project: To illustrate the hand application of LEM (Di = 9 [m])**
File: 2 Layers-Di=9m	**Date: 06-09-2017**
Page No.:	**Title: Double-layered soil (1 Sublayer)**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Double-layered soil (1 Sublayer)
Date: 06-09-2017
Project: To illustrate the hand application of LEM (Di = 0.9 [m])
File: 2 Layers-Di=0.9m

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 18.00
Depth increment in z-direction       Di          [m]     = 0.90

Time:
Time of consolidation                Tr          [Years] = 1.000
Time increment                       dT          [Years] = 0.100

Generation of times:
Start time                           To          [Years] = 1.000
No. of time intervals                Nt          [-]     = 2
Time interval                        Ti          [Years] = 1.000

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1       9.00         10      3.1710E-06      4.0000E-09
    2       9.00         10      7.9270E-07      1.0000E-09
------------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 86.45
Degree of consolidation        Us [%]  = 86.45
Settlement                     s  [cm] = 20.01
```

Initial and Final pore water pressures with depth:

No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
0	0.00	100.00	0.00
1	0.90	100.00	2.17
2	1.80	100.00	4.31
3	2.70	100.00	6.38
4	3.60	100.00	8.37
5	4.50	100.00	10.23
6	5.40	100.00	11.95
7	6.30	100.00	13.49
8	7.20	100.00	14.84
9	8.10	100.00	15.98
10	9.00	100.00	16.89
11	9.90	100.00	20.53
12	10.80	100.00	23.00
13	11.70	100.00	24.18
14	12.60	100.00	23.99
15	13.50	100.00	22.46
16	14.40	100.00	19.68
17	15.30	100.00	15.80
18	16.20	100.00	11.05
19	17.10	100.00	5.68
20	18.00	100.00	0.00

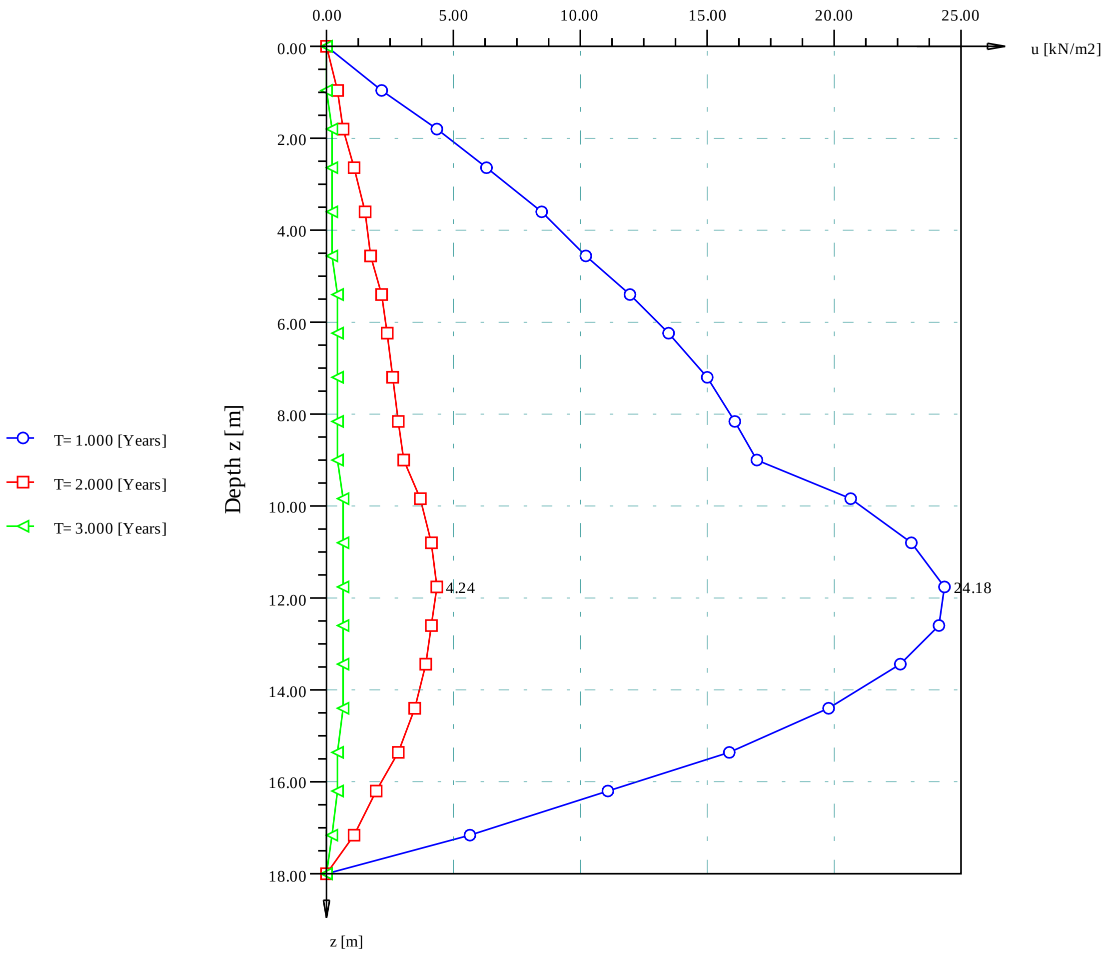
Pore water pressure U [kN/m2]:

T [Years] \ z [m]	1.000	2.000	3.000
0.00	0.00	0.00	0.00
0.90	2.17	0.38	0.07
1.80	4.31	0.75	0.13
2.70	6.38	1.11	0.19
3.60	8.37	1.45	0.25
4.50	10.23	1.77	0.31
5.40	11.95	2.07	0.36
6.30	13.49	2.34	0.41
7.20	14.84	2.58	0.45
8.10	15.98	2.78	0.49
9.00	16.89	2.94	0.51
9.90	20.53	3.58	0.63
10.80	23.00	4.03	0.70
11.70	24.18	4.24	0.74
12.60	23.99	4.22	0.74
13.50	22.46	3.95	0.69
14.40	19.68	3.47	0.61
15.30	15.80	2.79	0.49
16.20	11.05	1.95	0.34
17.10	5.68	1.00	0.18
18.00	0.00	0.00	0.00

Degree of consolidation/ Settlement:

T [Years]	1.000	2.000	3.000
Us [%]	86.45	97.63	99.58
s [cm]	20.01	22.60	23.05

Pore water pressure u [kN/m2]



Pore water pressure-Time

<table>
<tr><td colspan="2">**GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7**
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323</td></tr>
<tr><td>**Scale: 216**
File: 2 Layers-Di=0.9m
Page No.:</td><td>**Project: To illustrate the hand application of LEM (Di = 0.9 [m])**
Date: 06-09-2017
Title: Double-layered soil (1 Sublayer)</td></tr>
</table>

5.5.6 Example 5: Consolidation of a Multi-Layered Soil

5.5.6.1 Description of the problem:

Many authors applied their developed methods for analyzing the multi-layered system on the example presented by *Schiffiman* and *Stein* (1970). The example was studied by *Lee et al.* (1992) using a general analytical solution, by *Chen et al.* (2005) using finite difference method and differential quadrature method, by *Walker* (2006) using spectral method and by *Huang* and *Griffiths* (2010) using the finite-element method. The soil profile in the example consists of four layers. The geometry and the soil properties of the four layers are shown in Figure 5.47. The applied load on the layers was uniform.

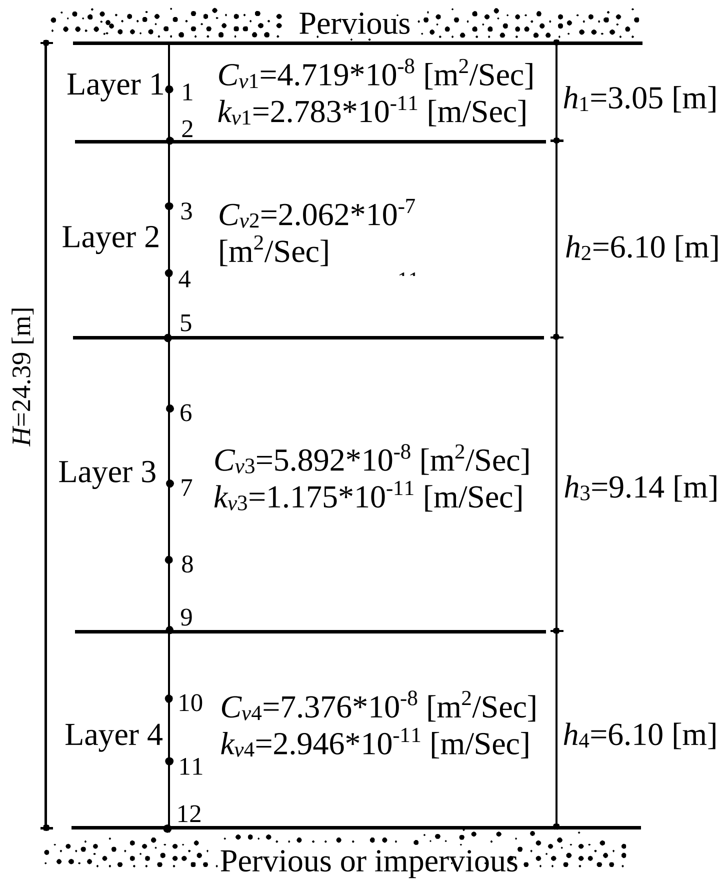


Figure 5.47 Four clay layers with studying nodes and soil properties

5.5.6.2 Analysis of the problem

To apply *LEM*, the four layers are divided into 2, 3, 4 and 3 sub-layers, respectively. The total grid nodes for the case of double drainage is N=11, while that for the case of single drainage is N=12. The case of double drainage that was studied by *Chen et al.* (2005) using differential quadrature method required to divide the soil profile into four differential quadrature elements with equally spaced sub-layers. The number of sampling nodes for the quadrature method of the four layers was taken as 7, 13, 19 and 13, respectively. This leads to a total sampling nodes of N=52. The sub-layer thickness was 0.51 [m]. In *LEM*, studying nodes are arbitrarily. Sub layers in *LEM* are 1.53 [m], 2.29 [m], 2.03 [m] and 2.29 [m], respectively. *Chen et al.* (2005) studied also the case of double drainage by finite difference method with the same sub-layer thickness of 0.51 [m]. The results of the quadrature method agreed well with the analytical solution while those of the finite difference had an obvious difference with the analytical solution.

5.5.6.3 Results:

Table 5.5 listed the Eigen values λ_j of the four layers obtained from the characteristic equation, (see mathematical model). However, λ_j of *LEM* are computed by a similar manner to that of *Lee et al.* (1992), they are different from those of *Lee et al.* (1992) for the same example. This is related to using a different depth ratio in *LEM*. In *LEM*, the number of λ_j is specified by the number of studying nodes, while that in the general analytical solution of *Lee et al.* (1992) is unlimited.

Figure 5.48 and Figure 5.49 show the excess pore water pressure ratio with depth ratio at different time factors obtained from *LEM* compared with that of the analytical solution presented by *Lee et al.* (1992) for both cases of single and double drainages. Figure 5.50 shows the degree of consolidation at different time factors $Tv=c_{v1}t/H^2$ for the double drainage case obtained from *LEM* compared with that of the differential quadrature method presented by *Chen et al.* (2005). It can be seen from those figures that results obtained by *LEM* are in a good agreement with those of *Lee et al.* (1992) and *Chen et al.* (2005).

Table 5.5 Eigen values of the four layers

Double drainage				Single drainage			
j	λ_j	j	λ_j	j	λ_j	j	λ_j
1	0.6309	7	3.6397	1	0.2524	7	3.4559
2	0.9367	8	4.0673	2	0.7216	8	3.9392
3	1.4444	9	4.6724	3	1.2326	9	4.2328
4	2.2204	10	5.2145	4	1.9140	10	4.8953
5	2.6107	11	5.7231	5	2.3810	11	5.6192
6	3.0565	-	-	6	2.7253	12	5.9248

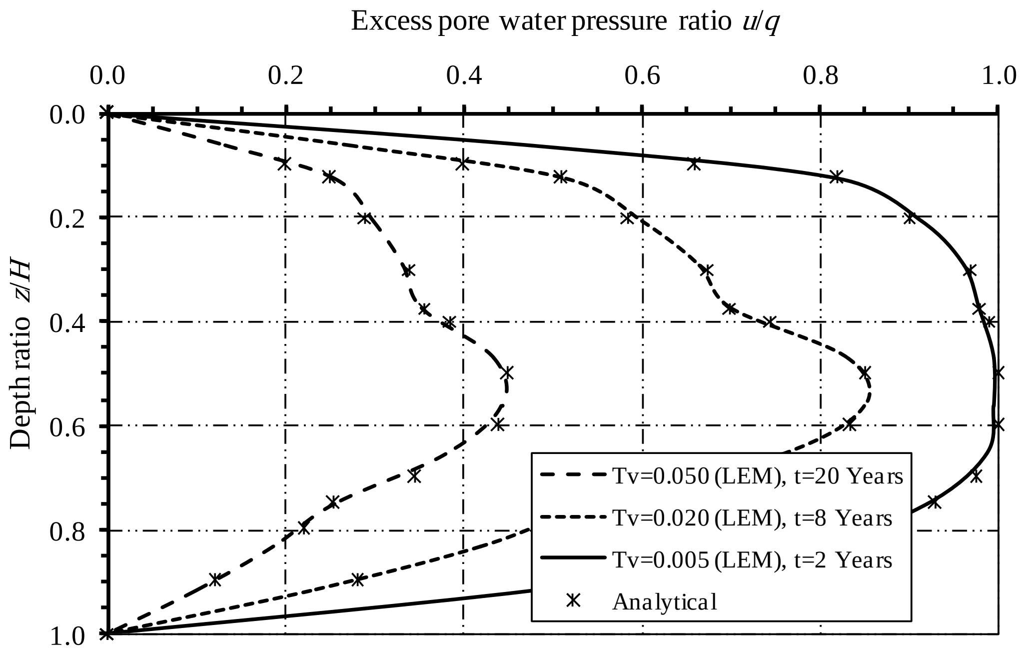


Figure 5.48 Excess pore water pressure ratio with depth ratio at different T_v for double drainage layer

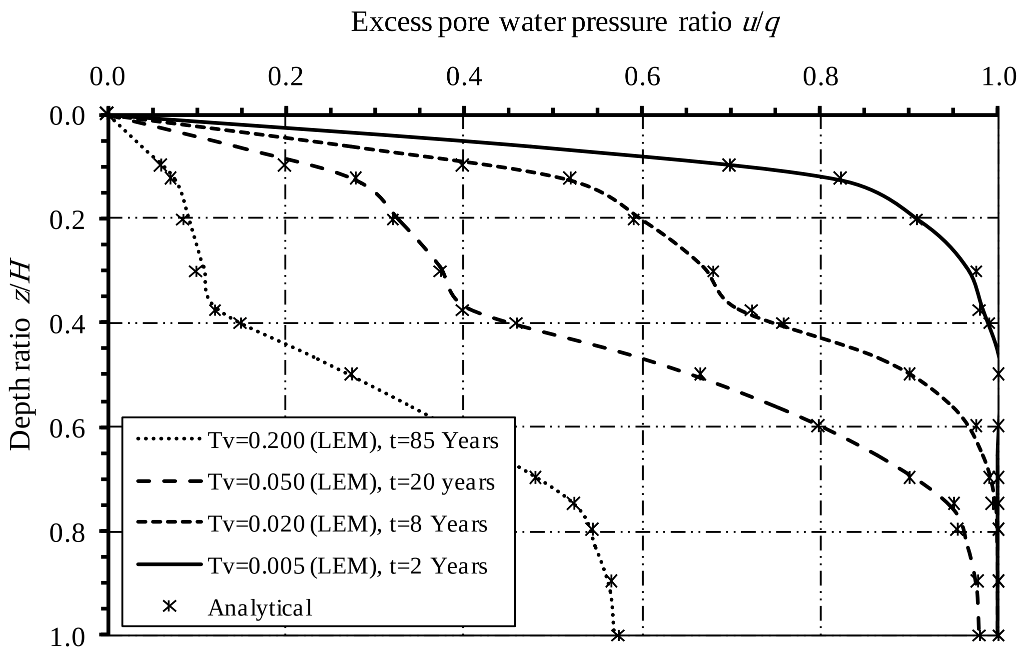


Figure 5.49 Excess pore water pressure ratio with depth ratio at different T_v for single drainage layer

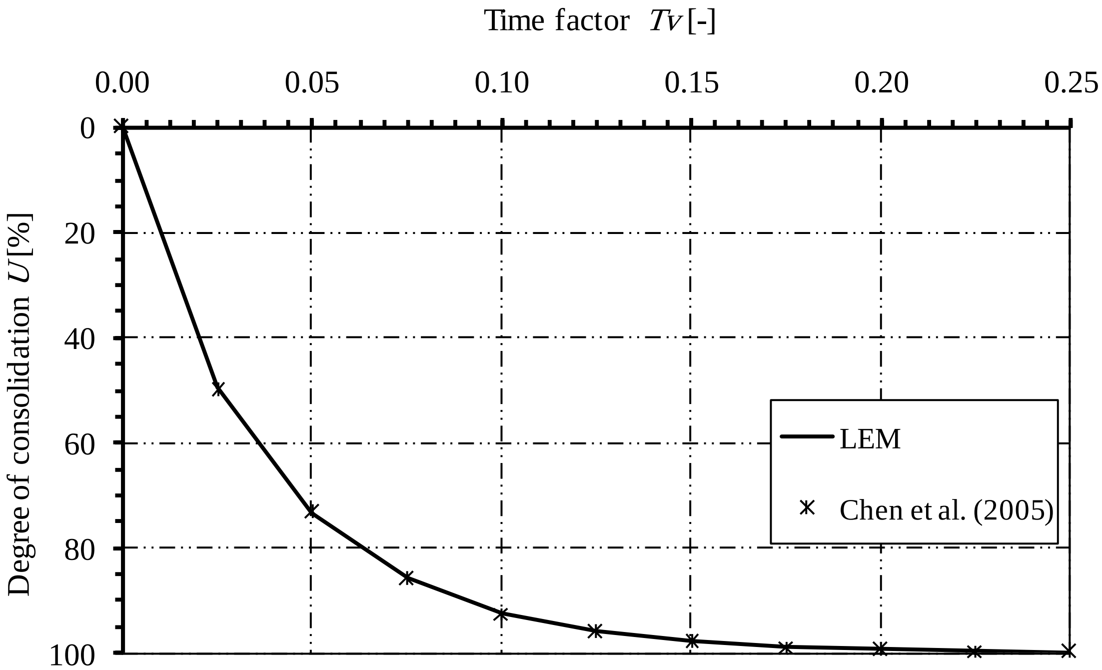


Figure 5.50 Degree of consolidation at different T_v

5.5.6.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation by *LEM* for both single and double drainage are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a multi-layered soil-(DD)
Date: 25_06_2015
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Four layers (DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Linear loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.40
Depth increment in z-direction       Di          [m]     = 1.53

Time:
Time of consolidation                Tr          [Years] = 2.000
Time of construction                 Tc          [Years] = 0.000

Generation of times:
Start time                           To          [Years] = 2.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              6.000
  2             12.000
----------------------

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No. thickness  sublayers   consolidation    permeability
    I         h        Nsl              Cv               k
  [-]       [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      3.10          3      4.7190E-08      2.7830E-11
    2      6.10          3      2.0620E-07      8.2550E-11
    3      9.10          8      5.8920E-08      1.1750E-11
    4      6.10          5      7.3760E-08      2.9460E-11
------------------------------------------------------------
```

Results:

Degree of consolidation	Up [%]	= 18.97
Degree of consolidation	Us [%]	= 25.26
Settlement	s [cm]	= 2.19

Initial and Final pore water pressures with depth:

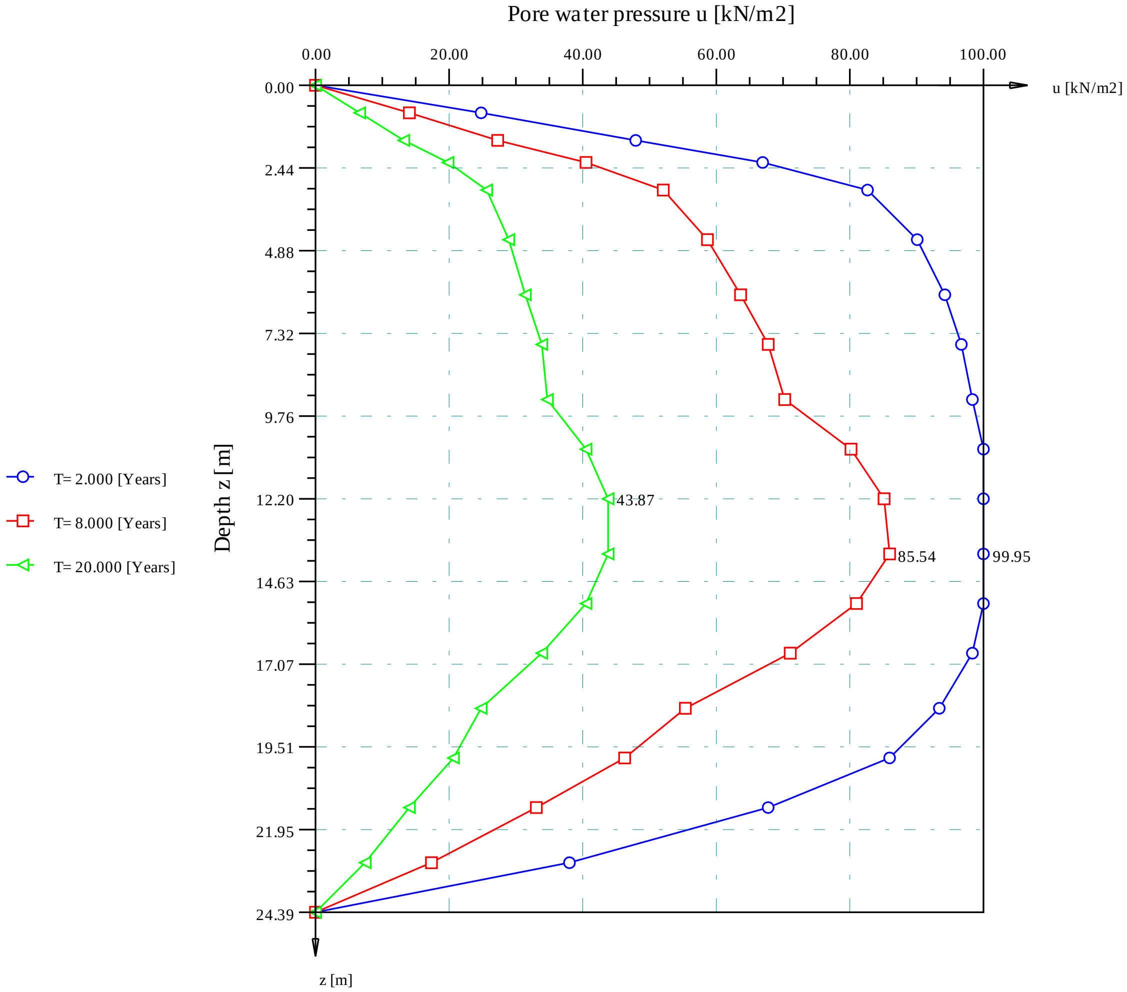
No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	0.78	100.00	24.97
2	1.55	100.00	47.86
3	2.33	100.00	67.23
4	3.10	100.00	82.55
5	4.63	100.00	89.86
6	6.15	100.00	94.45
7	7.68	100.00	96.98
8	9.20	100.00	98.03
9	10.72	100.00	99.62
10	12.23	100.00	99.94
11	13.75	100.00	99.95
12	15.27	100.00	99.71
13	16.78	100.00	98.40
14	18.30	100.00	93.31
15	19.83	100.00	85.59
16	21.35	100.00	67.49
17	22.88	100.00	37.78
18	24.40	100.00	0.00

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	2.000	8.000	20.000
0.00	0.00	0.00	0.00
0.78	24.97	13.85	6.70
1.55	47.86	27.32	13.24
2.33	67.23	40.07	19.46
3.10	82.55	51.77	25.23
4.63	89.86	58.50	28.59
6.15	94.45	63.91	31.36
7.68	96.98	67.91	33.47
9.20	98.03	70.44	34.88
10.72	99.62	80.42	40.85
12.23	99.94	85.40	43.87
13.75	99.95	85.54	43.69
15.27	99.71	80.79	40.31
16.78	98.40	70.93	33.94
18.30	93.31	55.72	25.03
19.83	85.59	46.48	20.29
21.35	67.49	33.43	14.28
22.88	37.78	17.49	7.37
24.40	0.00	0.00	0.00

Degree of consolidation/ Settlement:

T [Years]	2.000	8.000	20.000
Us [%]	25.26	50.54	76.12
s [cm]	2.19	4.39	6.61



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 826	**Project: A study on one-dimensional consolidation.... Lee (1992)- (DD)**
File: Four layers (DD)	**Date: 25_06_2015**
Page No.:	**Title: Consolidation of a multi-layered soil**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a multi-layered soil-(SD)
Date: 25_06_2015
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Four layers (SD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Linear loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.40
Depth increment in z-direction       Di          [m]     = 1.53

Time:
Time of consolidation                Tr          [Years] = 2.000
Time of construction                 Tc          [Years] = 0.000

Generation of times:
Start time                           To          [Years] = 2.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              6.000
  2             12.000
  3             65.000
----------------------
```

Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	3.10	3	4.7190E-08	2.7830E-11
2	6.10	3	2.0620E-07	8.2550E-11
3	9.10	8	5.8920E-08	1.1750E-11
4	6.10	5	7.3760E-08	2.9460E-11

Results:
Degree of consolidation Up [%] = 8.72
Degree of consolidation Us [%] = 13.70
Settlement s [cm] = 1.19

Initial and Final pore water pressures with depth:

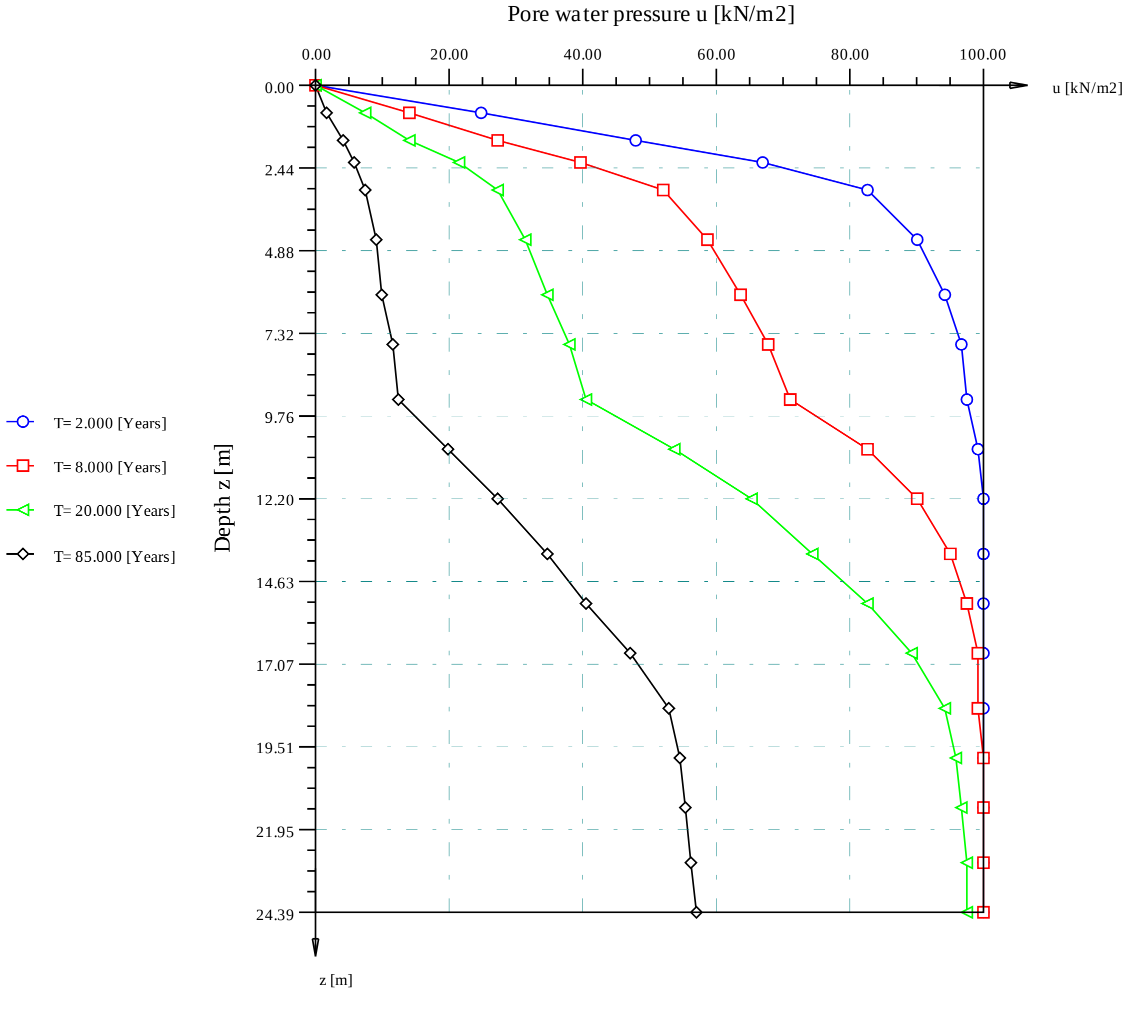
No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	0.78	100.00	24.97
2	1.55	100.00	47.86
3	2.33	100.00	67.23
4	3.10	100.00	82.55
5	4.63	100.00	89.86
6	6.15	100.00	94.45
7	7.68	100.00	96.99
8	9.20	100.00	98.03
9	10.72	100.00	99.62
10	12.23	100.00	99.94
11	13.75	100.00	99.99
12	15.27	100.00	100.00
13	16.78	100.00	100.00
14	18.30	100.00	100.00
15	19.83	100.00	100.00
16	21.35	100.00	100.00
17	22.88	100.00	100.00
18	24.40	100.00	100.01

Pore water pressure U [kN/m2]:

T [Years] \ z [m]	2.000	8.000	20.000	85.000
0.00	0.00	0.00	0.00	0.00
0.78	24.97	13.86	7.24	1.94
1.55	47.86	27.34	14.34	3.88
2.33	67.23	40.11	21.19	5.80
3.10	82.55	51.85	27.66	7.69
4.63	89.86	58.62	31.58	8.93
6.15	94.45	64.09	35.01	10.12
7.68	96.99	68.20	37.92	11.29
9.20	98.03	70.90	40.28	12.41
10.72	99.62	82.53	53.79	20.00
12.23	99.94	90.17	65.29	27.34
13.75	99.99	94.83	74.84	34.35
15.27	100.00	97.48	82.65	40.95
16.78	100.00	98.91	89.01	47.06
18.30	100.00	99.69	94.23	52.61
19.83	100.00	99.87	95.84	54.46
21.35	100.00	99.95	96.92	55.80
22.88	100.00	99.98	97.54	56.60
24.40	100.01	99.99	97.73	56.87

Degree of consolidation/ Settlement:

T [Years]	2.000	8.000	20.000	85.000
Us [%]	13.70	28.03	44.01	73.24
s [cm]	1.19	2.43	3.82	6.36



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 828	**Project: A study on one-dimensional consolidation.... Lee (1992)- (SD)**
File: Four layers (SD)	**Date: 25_06_2015**
Page No.:	**Title: Consolidation of a multi-layered soil**

5.5.7 Example 6: Consolidation of a Multi-Layered Soil for High Values of *Tv*

5.5.7.1 Description of the problem:

The test Example 5 is analyzed for high values of *Tv*. Results of *LEM* in this case are compared with those of *FDM*. *Tv* is chosen to be 0.1, 0.15 and 0.25 for pervious bottom boundary and 0.25, 0.3 and 0.5 for impervious bottom boundary. The below results of *LEM* show a good agreement with those of *FDM* even for high values of *Tv*.

5.5.7.2 Data:

Initial pore water pressure	*uo*	[kN/m^2]	= 100.0
Total layer thickness	H_d	[m]	= 24.39
Depth increment in *z*-direction	*Di*	[m]	= 0.20

Table 5.6 Soil data

Layer No.	Layer Thickness *h* [m]	*LEM*		*FDM*		Coefficient of consolidation *Cv* [m^2/s]	Coefficient of permeability *K* [m/s]
		No. of sub layers	sub layer thickness [m]	No. of sub layers	sub layer thickness [m]		
1	3.05	16	0.19	16	0.19	4.719×10^{-8}	2.783×10^{-11}
2	6.10	16	0.19	32	0.19	2.062×10^{-7}	8.255×10^{-11}
3	9.14	43	0.21	46	0.20	5.892×10^{-8}	1.175×10^{-11}
4	6.10	26	0.23	32	0.19	7.376×10^{-8}	2.946×10^{-11}

5.5.7.3 Results:

Results of *LEM* are compared with those of *FDM* in Table 5.7 and Table 5.8, Figure 5.51 and Figure 5.52 at different time factors. One can see that results obtained from *LEM* are in a good agreement with those of *FDM* for both cases of double and single drainage.

Table 5.7 Comparison of results for pervious bottom boundary

Time factor/ Years	*LEM*		*FDM*	
	UP %	US %	UP %	US %
Tv=0.10/ *t*=43 Years	93.14	94.02	93.62	94.44
Tv=0.15/ *t*=64 Years	98.05	98.30	98.25	98.47
Tv=0.25/ *t*=107 Years	99.85	99.87	99.88	99.89

Table 5.8 Comparison of results for impervious bottom boundary

Time factor/ Years	*LEM*		*FDM*	
	UP %	US %	UP %	US %
Tv=0.25/ *t*=107 Years	76.16	78.32	76.64	78.76
Tv=0.30/ *t*=128 Years	80.49	82.26	80.95	82.68
Tv=0.50/ *t*=211 Years	91.17	91.97	91.49	92.27

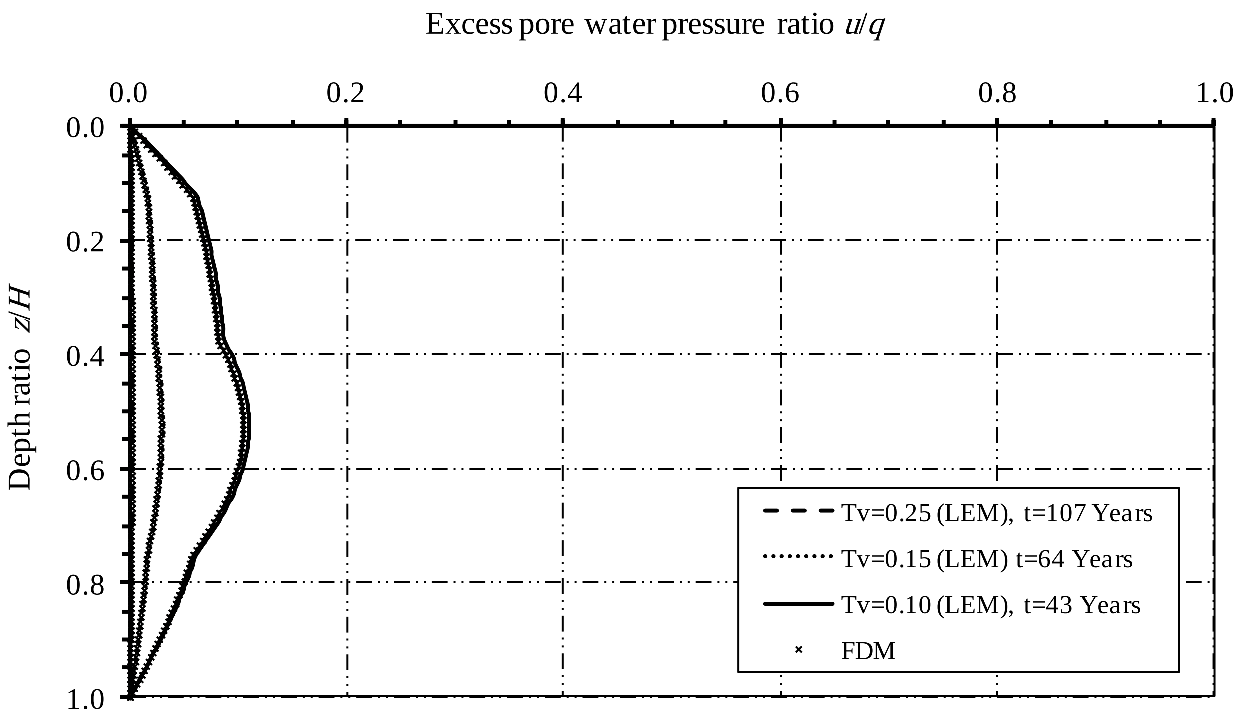


Figure 5.51 Excess pore water pressure ratio with depth ratio at different *Tv* (double drainage)

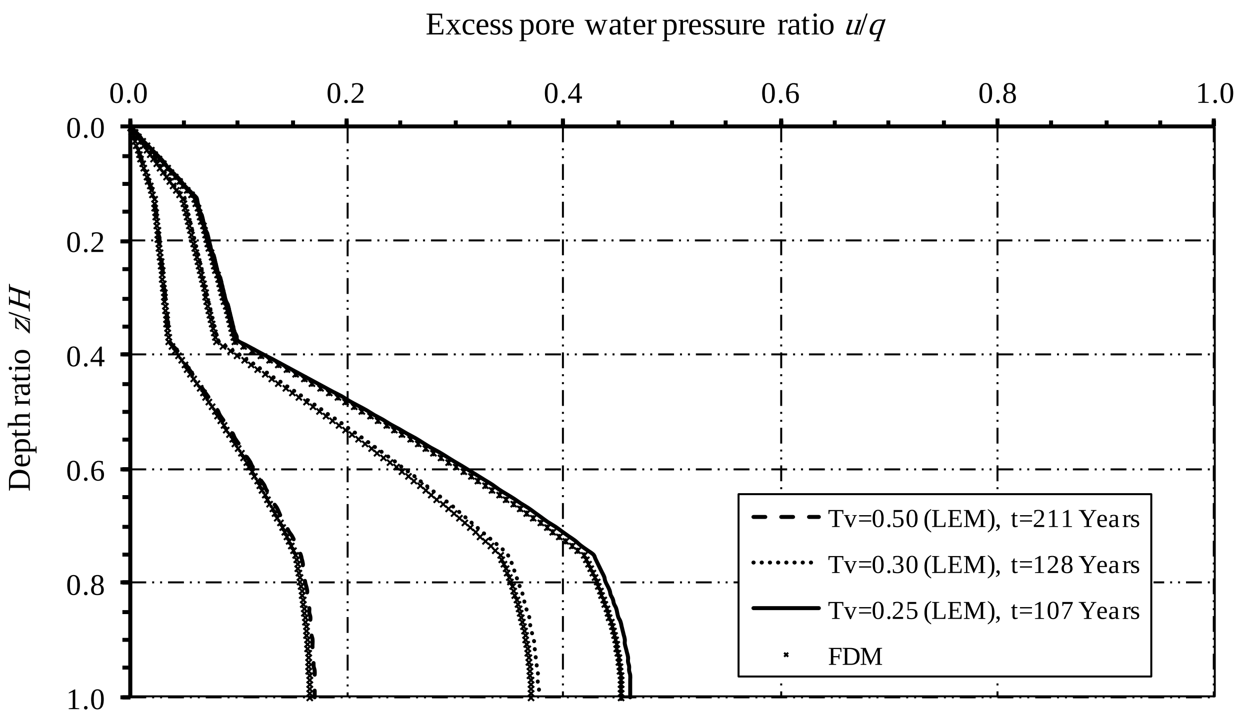


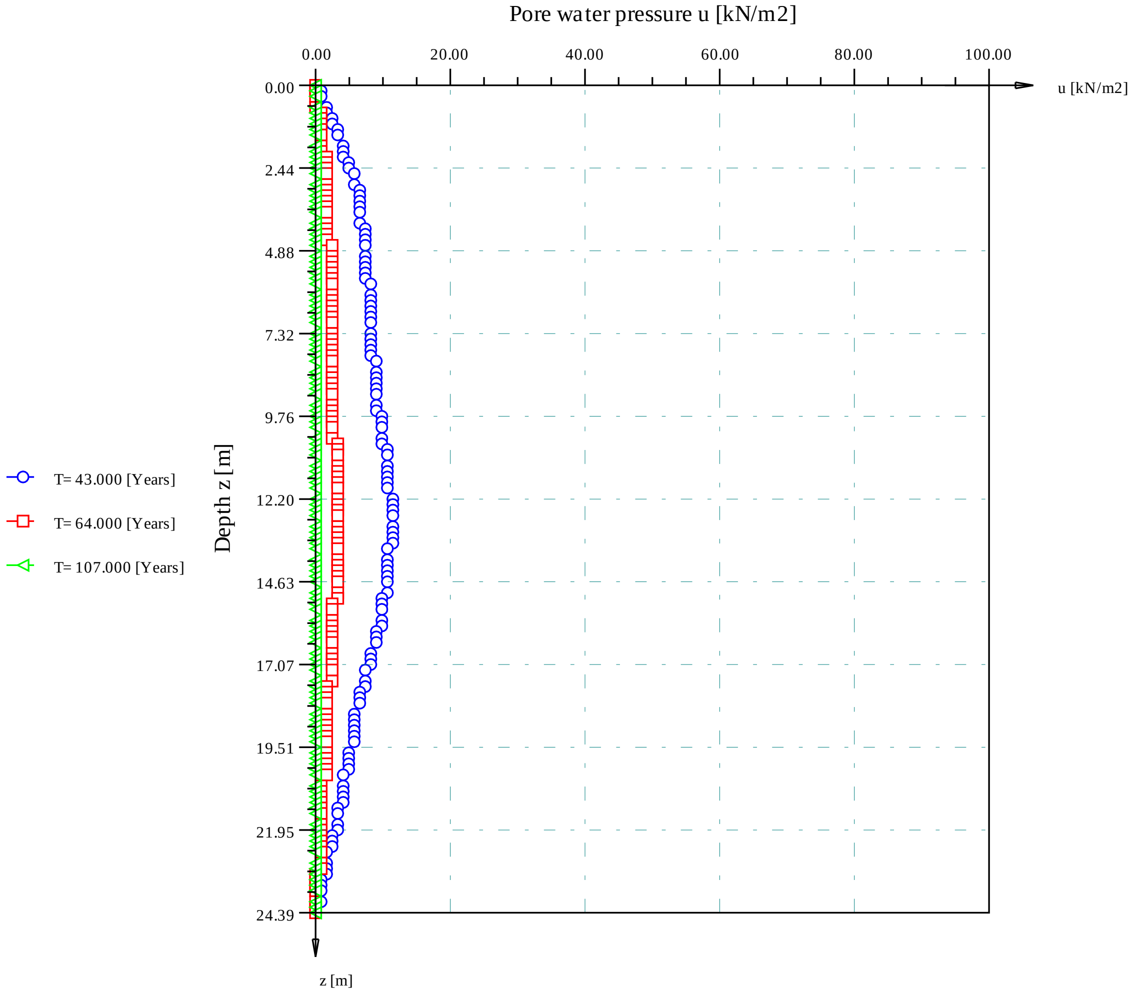
Figure 5.52 Excess pore water pressure ratio with depth ratio at different *Tv* (single drainage)

5.5.7.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation by *LEM* and *FDM* are presented on the next pages.

```
*************************************************************
                         GEO Tools
                          Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a Multi-Layered Soil for High Values of Tv (DD)
Date: 07-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: 4 layers High Tv (DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.39
Depth increment in z-direction       Di          [m]     = 0.20

Time:
Time of consolidation                Tr          [Years] = 43.000

Generation of times:
Start time                           To          [Years] = 43.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1             21.000
  2             43.000
----------------------

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1       3.05         16      4.7190E-08      2.7830E-11
    2       6.10         16      2.0620E-07      8.2550E-11
    3       9.14         43      5.8920E-08      1.1750E-11
    4       6.10         26      7.3760E-08      2.9460E-11
------------------------------------------------------------
```

```
Results:
Degree of consolidation        Up [%]  = 93.14
Degree of consolidation        Us [%]  = 94.02
Settlement                     s  [cm] = 8.15


Degree of consolidation/ Settlement:
----------------------------------------------------

T [Years]      43.000      64.000      107.000

----------------------------------------------------
  Us  [%]       94.02       98.30        99.87
   s [cm]        8.15        8.52         8.65
----------------------------------------------------
```

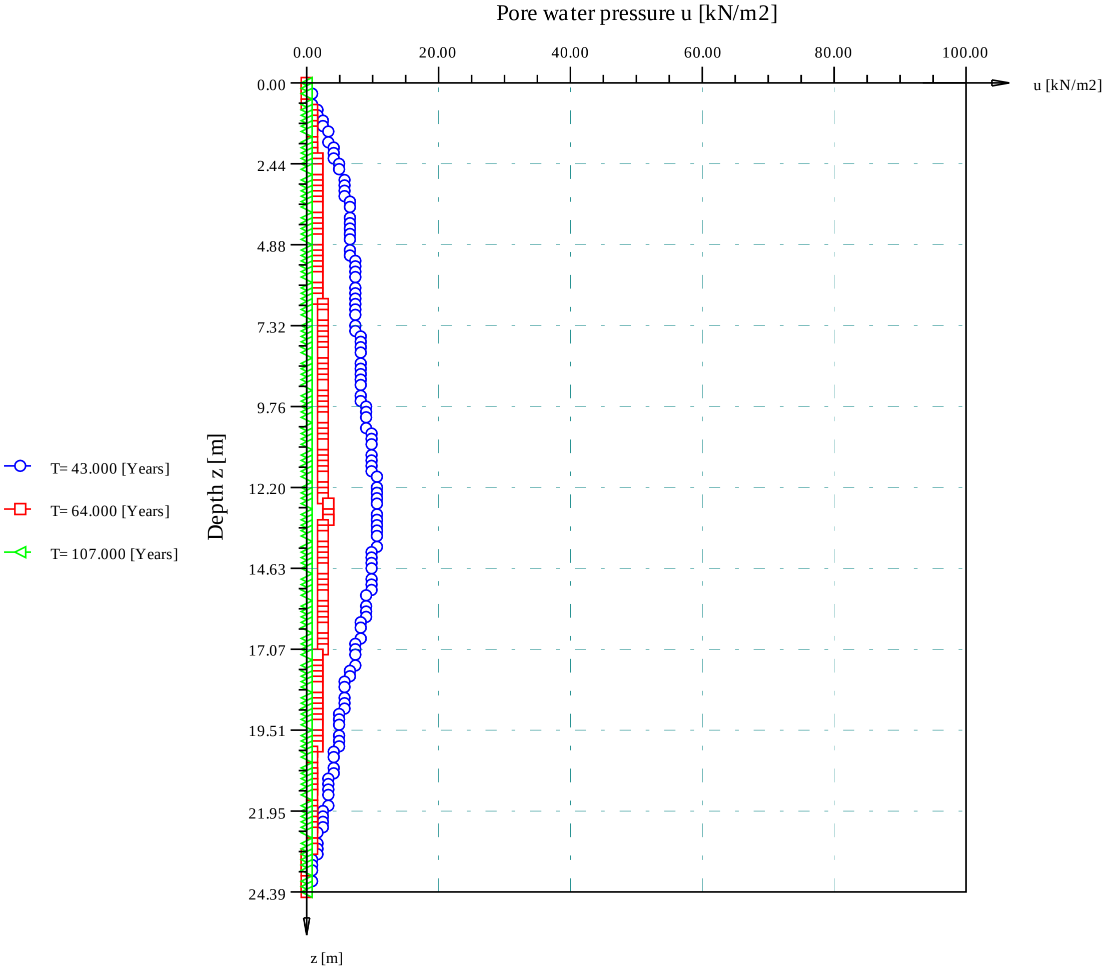


Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 817	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: 4 layers High Tv (DD)	**Date: 07-09-2017**
Page No.:	**Title: Consolidation of a Multi-Layered Soil for High Values of Tv (DD)**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a Multi-Layered Soil for High Values of Tv (DD)
Date: 07-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: 4 layers High Tv (DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.39
Depth increment in z-direction       Di          [m]     = 0.20

Time:
Time of consolidation                Tr          [Years] = 43.000
Time increment                       dT          [Years] = 0.250

Generation of times:
Start time                           To          [Years] = 43.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1             21.000
  2             43.000
----------------------

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1       3.05         16      4.7190E-08      2.7830E-11
    2       6.10         32      2.0620E-07      8.2550E-11
    3       9.14         46      5.8920E-08      1.1750E-11
    4       6.10         32      7.3760E-08      2.9460E-11
------------------------------------------------------------
```

```
Results:
Degree of consolidation        Up [%]  = 93.62
Degree of consolidation        Us [%]  = 94.44
Settlement                     s  [cm] = 8.18

Degree of consolidation/ Settlement:
-----------------------------------------------------

T [Years]      43.000      64.000      107.000

-----------------------------------------------------
  Us  [%]       94.44       98.47        99.89
   s [cm]        8.18        8.53         8.66
-----------------------------------------------------
```

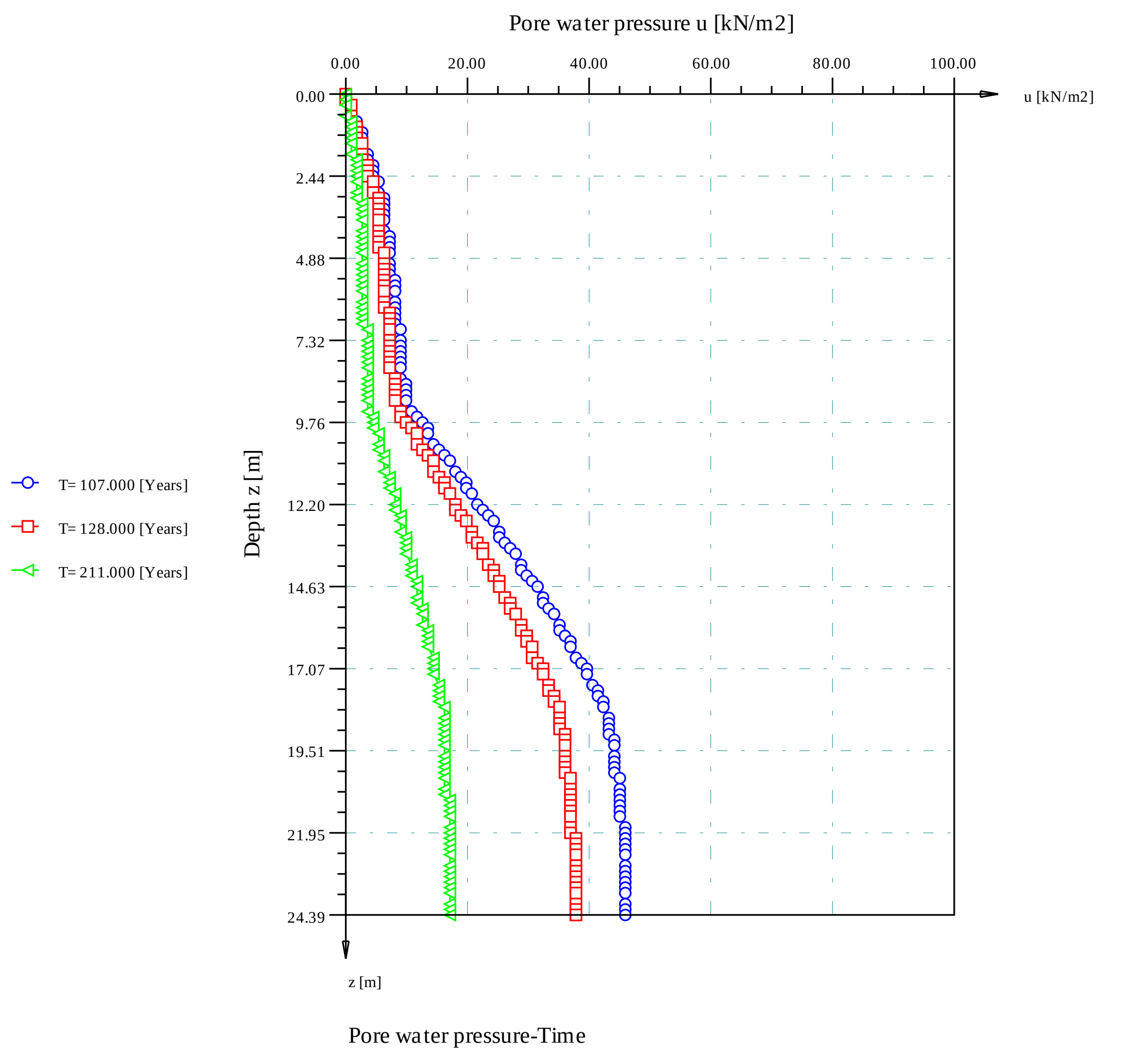


Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 817	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: 4 layers High Tv (DD)	**Date: 07-09-2017**
Page No.:	**Title: Consolidation of a Multi-Layered Soil for High Values of Tv (DD)**

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a Multi-Layered Soil for High Values of Tv (SD)
Date: 07-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: 4 layers High Tv (SD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure        uo          [kN/m2] = 100.00
Overburden pressure                 Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                     Hb          [m]     = 24.39
Depth increment in z-direction      Di          [m]     = 0.20

Time:
Time of consolidation               Tr          [Years] = 107.000

Generation of times:
Start time                          To          [Years] = 107.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1             21.000
  2             83.000
----------------------

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1       3.05         16      4.7190E-08      2.7830E-11
    2       6.10         16      2.0620E-07      8.2550E-11
    3       9.14         43      5.8920E-08      1.1750E-11
    4       6.10         26      7.3760E-08      2.9460E-11
-----------------------------------------------------------
```

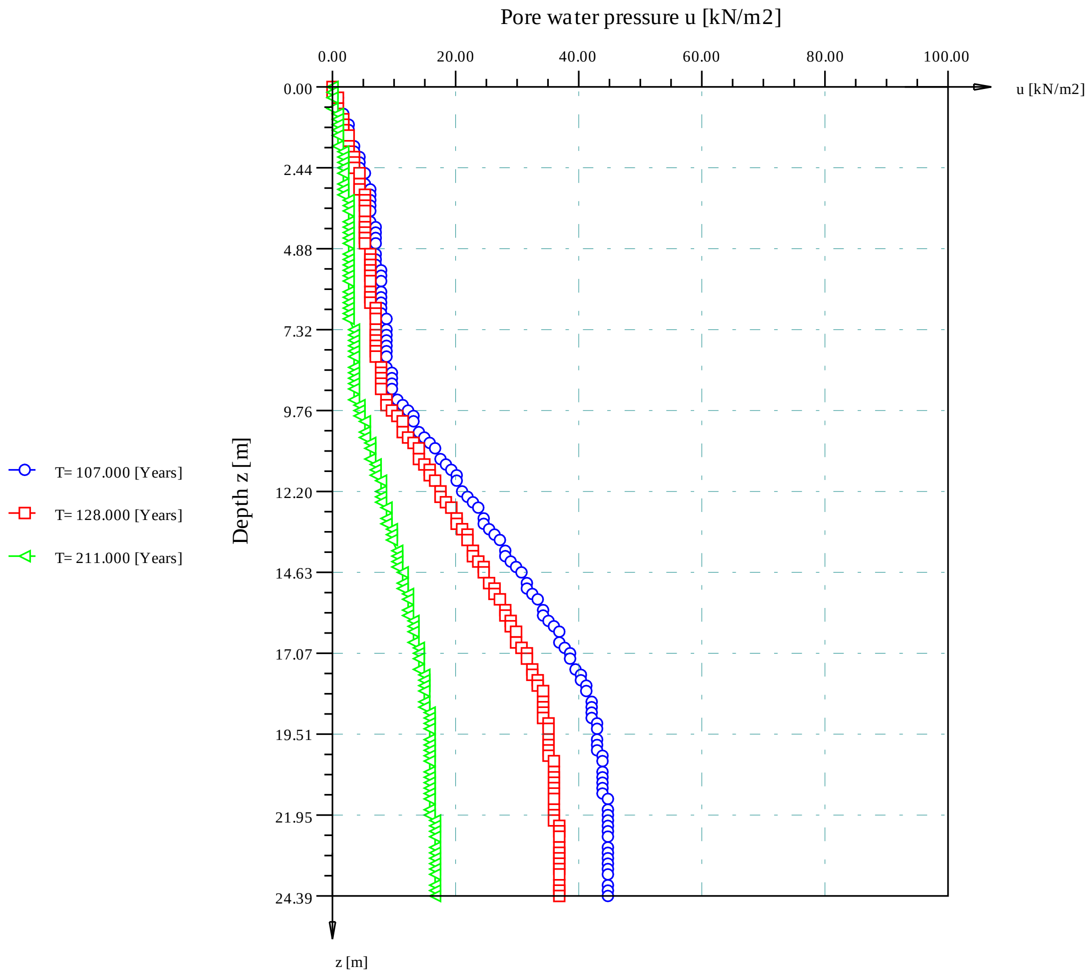
```
Results:
Degree of consolidation        Up [%]  = 76.16
Degree of consolidation        Us [%]  = 78.32
Settlement                     s  [cm] = 6.79


Degree of consolidation/ Settlement:
------------------------------------------------------

T [Years]      107.000      128.000      211.000

------------------------------------------------------
  Us  [%]        78.32        82.26        91.97
   s [cm]         6.79         7.13         7.97
------------------------------------------------------
```



GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 900	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: 4 layers High Tv (SD)	**Date: 07-09-2017**
Page No.:	**Title: Consolidation of a Multi-Layered Soil for High Values of Tv (SD)**

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a Multi-Layered Soil for High Values of Tv (SD)
Date: 07-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: 4 layers High Tv (SD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure        uo          [kN/m2] = 100.00
Overburden pressure                 Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                     Hb          [m]     = 24.39
Depth increment in z-direction      Di          [m]     = 0.20

Time:
Time of consolidation               Tr          [Years] = 107.000
Time increment                      dT          [Years] = 0.250

Generation of times:
Start time                          To          [Years] = 107.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1             21.000
  2             83.000
----------------------

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1       3.05         16      4.7190E-08      2.7830E-11
    2       6.10         32      2.0620E-07      8.2550E-11
    3       9.14         46      5.8920E-08      1.1750E-11
    4       6.10         32      7.3760E-08      2.9460E-11
-----------------------------------------------------------
```

```
Results:
Degree of consolidation        Up [%]  = 76.64
Degree of consolidation        Us [%]  = 78.76
Settlement                     s  [cm] = 6.82

Degree of consolidation/ Settlement:
-------------------------------------------------------

T [Years]      107.000      128.000      211.000

-------------------------------------------------------
  Us  [%]        78.76        82.68        92.27
   s [cm]         6.82         7.16         7.99
-------------------------------------------------------
```



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 880	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: 4 layers High Tv (SD)	**Date: 07-09-2017**
Page No.:	**Title: Consolidation of a Multi-Layered Soil for High Values of Tv (SD)**

5.5.8 Example 7: Consolidation of a Equivalent Layer for High Values of *Tv*

5.5.8.1 Description of the problem:

An "equivalent" soil layer having the same depth and the highest *Cv* value (i.e. 2.062 x 10×10^{-7}) for the Test Example 5 is also tested for times 23, 46 and 69 years. The layer is divided into 122 sub layers, each 0.2 m. Both below results of *LEM* and *FDM* are identical.

5.5.8.2 Data:

Initial pore water pressure	*uo*	[kN/m^2]	= 100.0
Total layer thickness	H_d	[m]	= 24.39
Depth increment in z-direction	*Di*	[m]	= 0.20

Table 5.9 Soil data

Layer No.	Layer Thickness *h* [m]	No. of sub layers [-]	sub layer thickness [m]	Coefficient of consolidation *Cv* [m^2/s]	Coefficient of permeability *K* [m/s]
1	24.39	122	0.20	2.062×10^{-7}	8.255×10^{-11}

5.5.8.3 Results:

Results of *LEM* are compared with those of *FDM* in Table 5.10 and Table 5.8Table 5.11, Figure 5.53 and Figure 5.52Figure 5.54 at different times. One can see that results obtained from *LEM* are in a good agreement with those of *FDM* for both cases of double and single drainage.

Table 5.10 Comparison of results for pervious bottom boundary

Time [Years]	*LEM*		*FDM*	
	UP %	US %	UP %	US %
t=23 Years	93.22	93.22	93.22	93.22
t=46 Years	99.43	99.43	99.43	99.43
t=69 Years	99.95	99.95	99.95	99.95

Table 5.11 Comparison of results for impervious bottom boundary

Time [Years]	*LEM*		*FDM*	
	UP %	US %	UP %	US %
t=23 Years	56.38	56.38	56.38	56.38
t=46 Years	76.56	76.56	76.56	76.56
t=69 Years	87.40	87.40	87.39	87.39

5.5.8.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation by *LEM* and *FDM* are presented on the next pages.

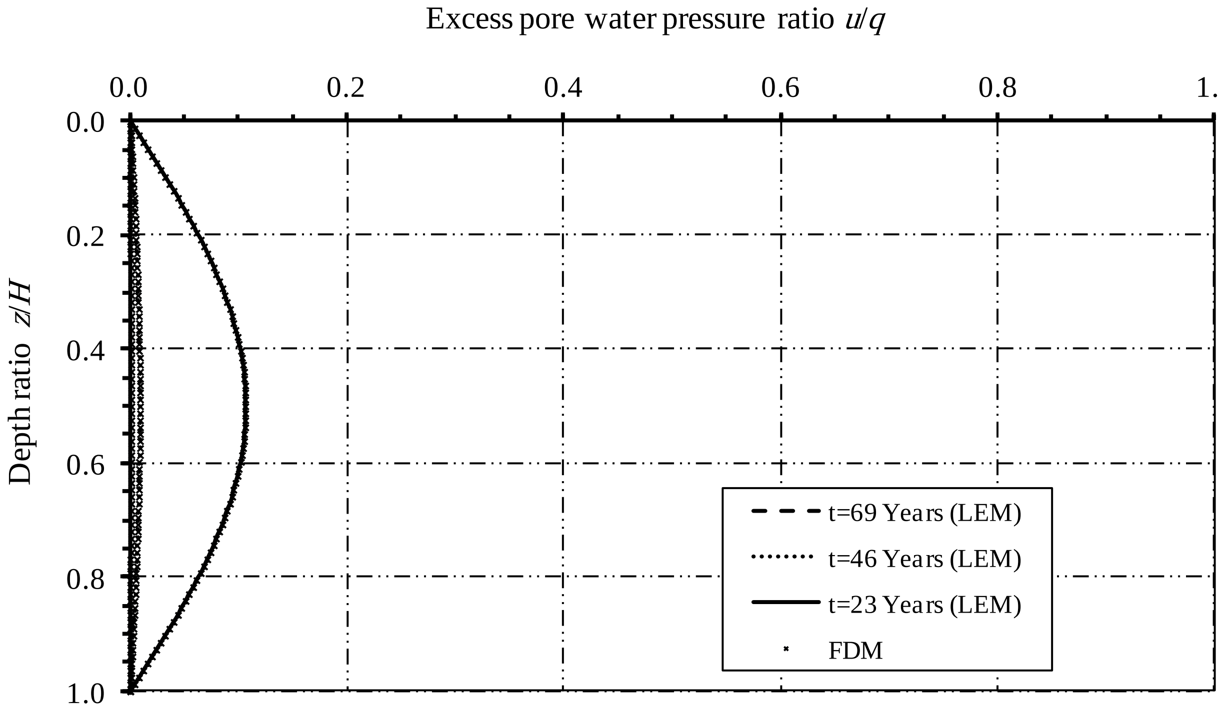


Figure 5.53 Excess pore water pressure ratio with depth ratio at different t (double drainage)

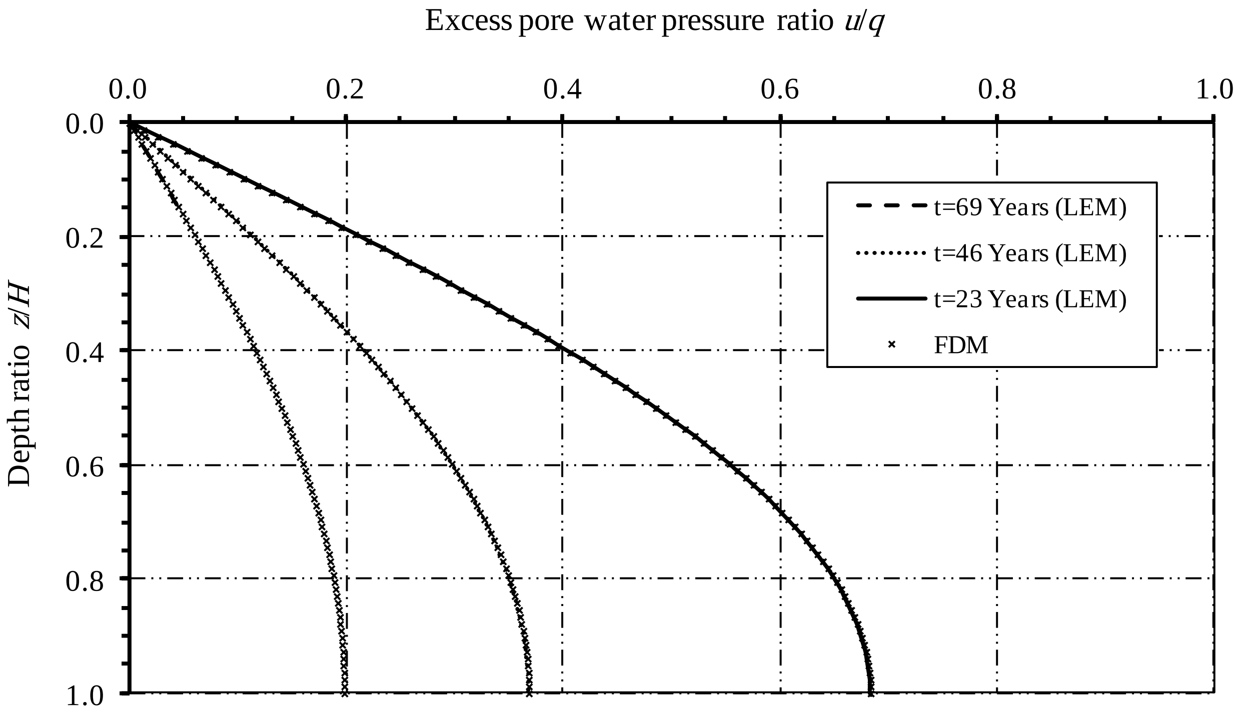
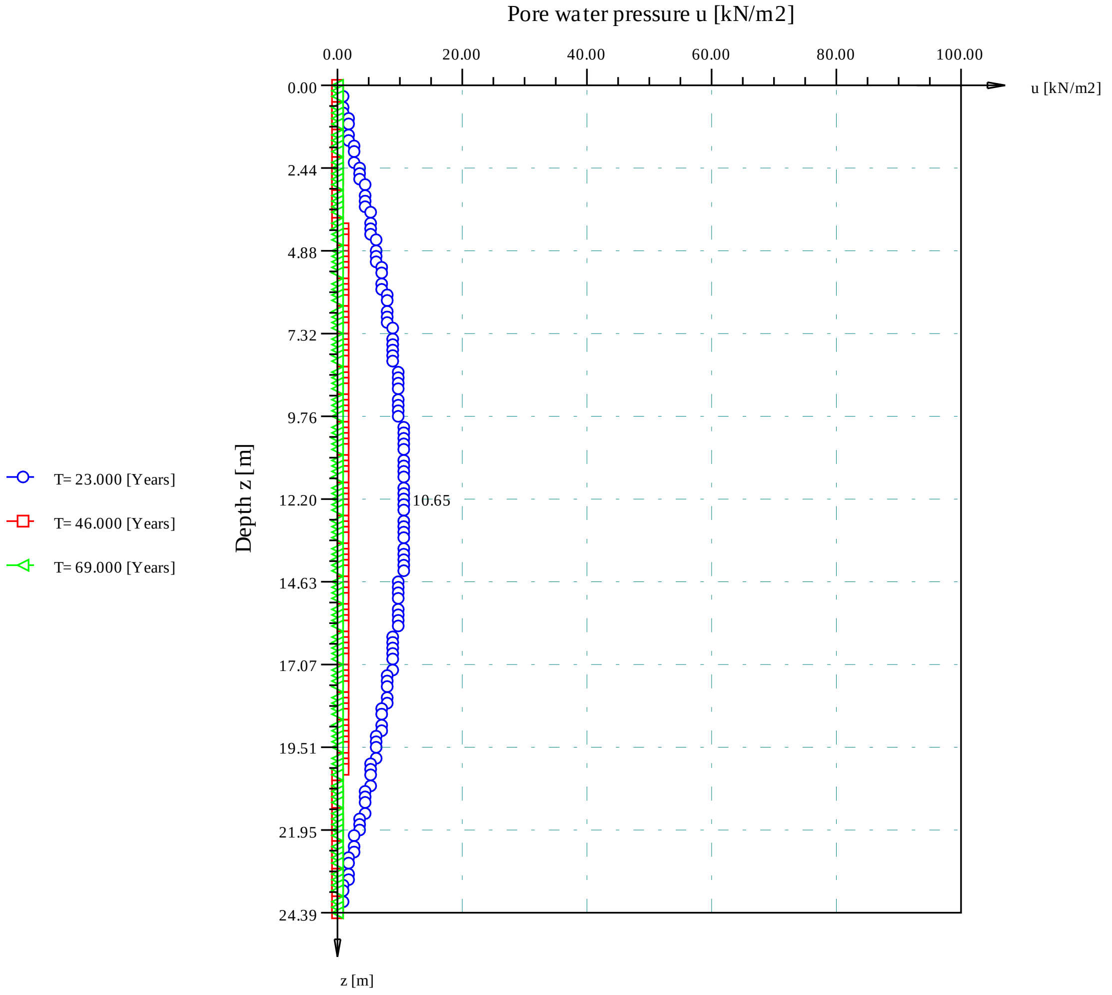


Figure 5.54 Excess pore water pressure ratio with depth ratio at different t (single drainage)

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)
Date: 14-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Equivalent layer (DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.39
Depth increment in z-direction       Di          [m]     = 0.20

Time:
Time of consolidation                Tr          [Years] = 23.000

Generation of times:
Start time                           To          [Years] = 23.000
No. of time intervals                Nt          [-]     = 2
Time interval                        Ti          [Years] = 23.000

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      24.39        122      2.0620E-07      8.2550E-11
------------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 93.22
Degree of consolidation        Us [%]  = 93.22
Settlement                     s  [cm] = 9.28
```

```
Degree of consolidation/ Settlement:
---------------------------------------------------

T [Years]      23.000      46.000      69.000

---------------------------------------------------
  Us  [%]       93.22       99.43       99.95
   s [cm]        9.28        9.90        9.95
---------------------------------------------------
```



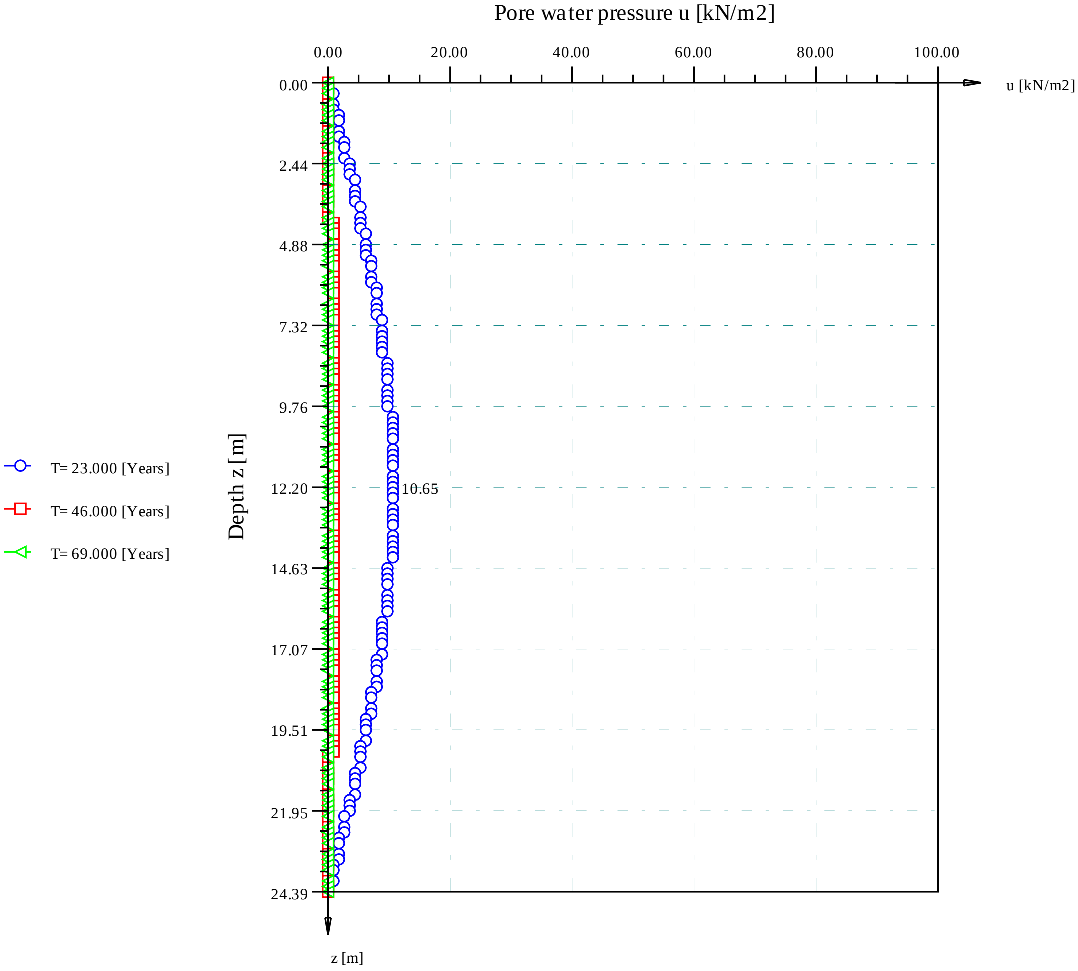
Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 883	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: Equivalent layer (DD)	**Date: 14-09-2017**
Page No.:	**Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)
Date: 14-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Equivalent layer (DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.39
Depth increment in z-direction       Di          [m]     = 0.20

Time:
Time of consolidation                Tr          [Years] = 23.000
Time increment                       dT          [Years] = 0.250

Generation of times:
Start time                           To          [Years] = 23.000
No. of time intervals                Nt          [-]     = 2
Time interval                        Ti          [Years] = 23.000

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      24.39        122      2.0620E-07      8.2550E-11
------------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 93.22
Degree of consolidation        Us [%]  = 93.22
Settlement                     s  [cm] = 9.28
```

```
Degree of consolidation/ Settlement:
---------------------------------------------------

T [Years]      23.000      46.000      69.000

---------------------------------------------------
  Us  [%]       93.22       99.43       99.95
   s [cm]        9.28        9.90        9.95
---------------------------------------------------
```

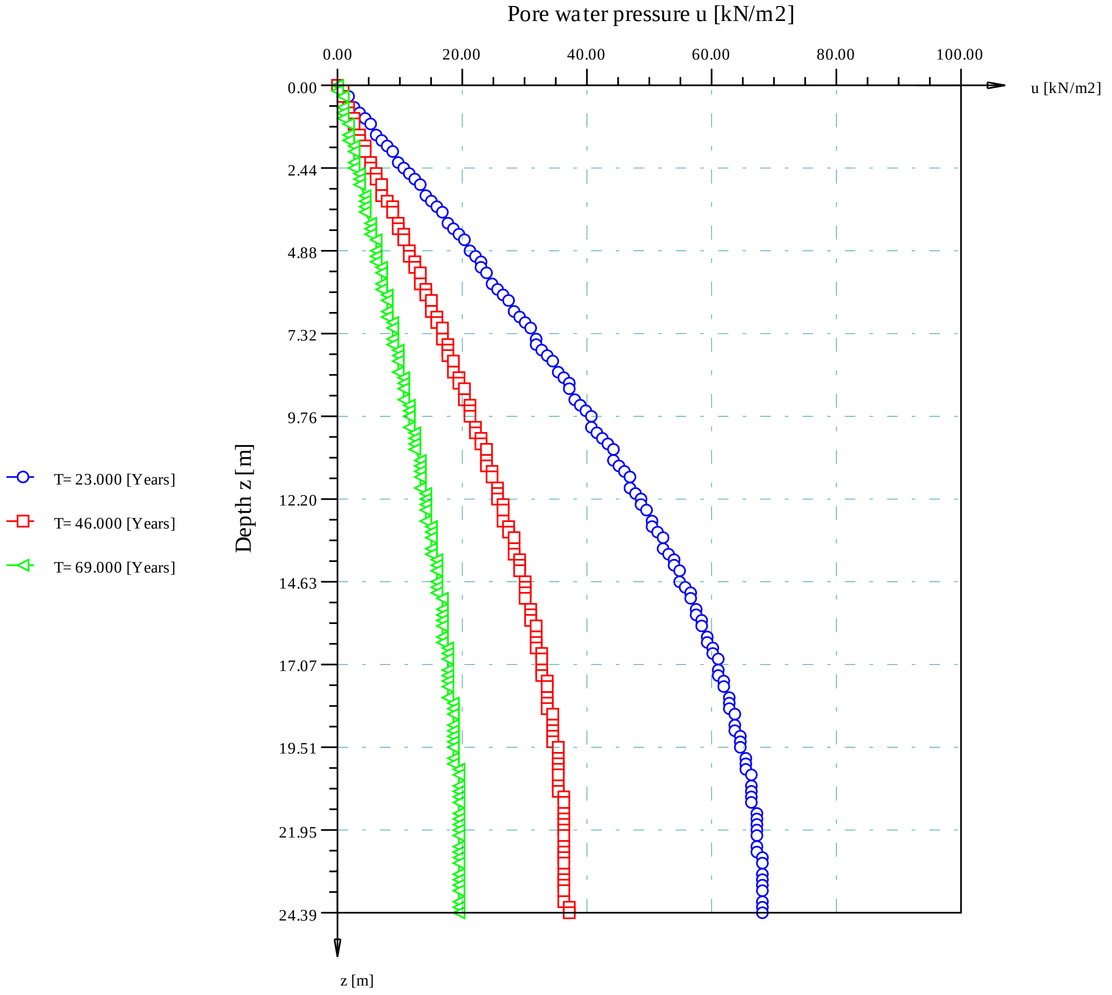


Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 **PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323**	
Scale: 883	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: Equivalent layer (DD)	**Date: 14-09-2017**
Page No.:	**Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)**

```
**********************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
**********************************************************
Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)
Date: 14-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Equivalent layer (SD)

----------------------------------------------------------
Degree of consolidation
----------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo         [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 24.39
Depth increment in z-direction       Di         [m]     = 0.20

Time:
Time of consolidation                Tr         [Years] = 23.000

Generation of times:
Start time                           To         [Years] = 23.000
No. of time intervals                Nt         [-]     = 2
Time interval                        Ti         [Years] = 23.000

Boring:
---------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
---------------------------------------------------------
    1      24.39        122      2.0620E-07      8.2550E-11
---------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 56.38
Degree of consolidation        Us [%]  = 56.38
Settlement                     s  [cm] = 5.61
```

```
Degree of consolidation/ Settlement:
---------------------------------------------------

T [Years]      23.000      46.000      69.000

---------------------------------------------------
  Us  [%]       56.38       76.56       87.40
   s [cm]        5.61        7.62        8.70
---------------------------------------------------
```

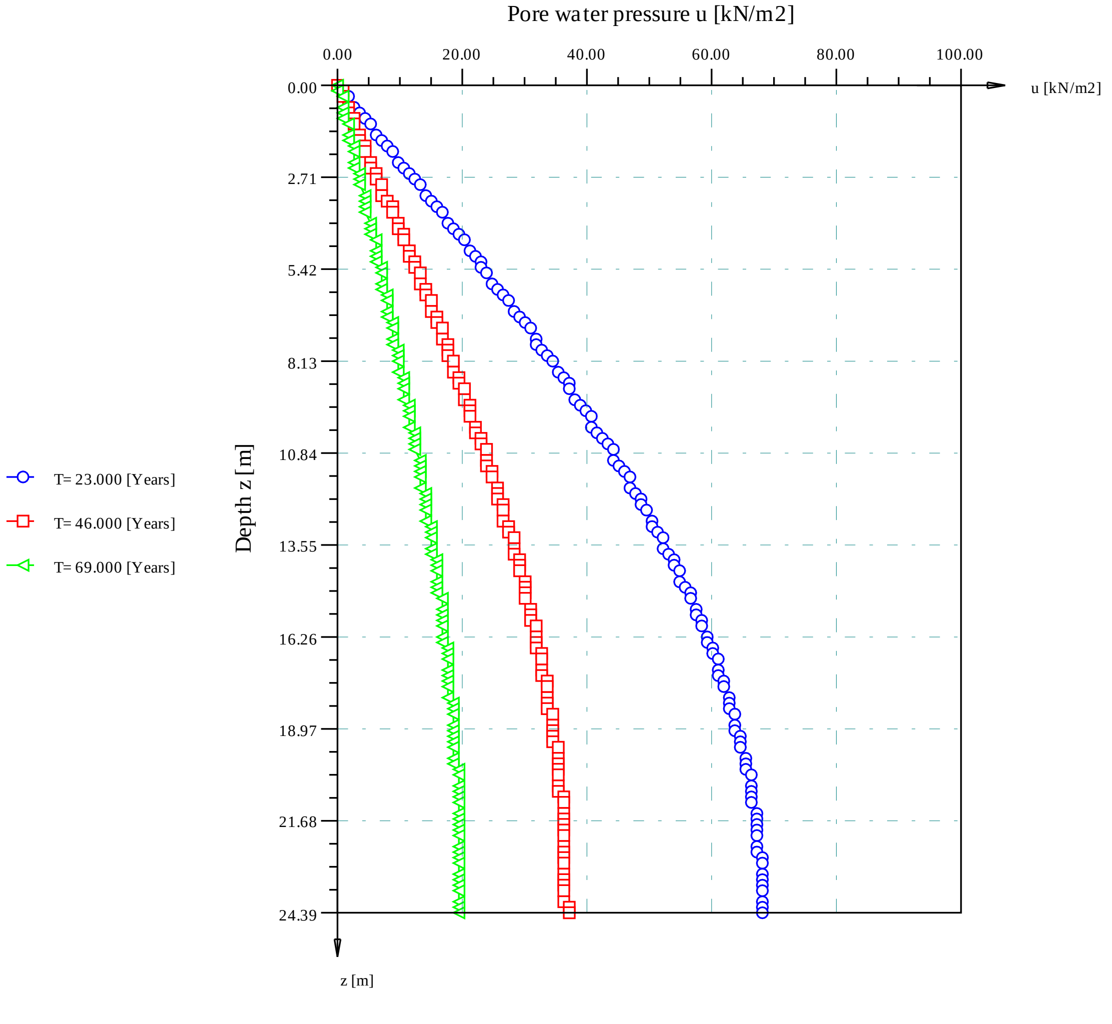


Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 887	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: Equivalent layer (SD)	**Date: 14-09-2017**
Page No.:	**Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)**

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)
Date: 14-09-2017
Project: A study on one-dimensional consolidation.... Lee (1992)
File: Equivalent layer (SD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 24.39
Depth increment in z-direction       Di          [m]     = 0.20

Time:
Time of consolidation                Tr          [Years] = 23.000
Time increment                       dT          [Years] = 0.250

Generation of times:
Start time                           To          [Years] = 23.000
No. of time intervals                Nt          [-]     = 2
Time interval                        Ti          [Years] = 23.000

Boring:
-----------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
-----------------------------------------------------------
    1      24.39        122      2.0620E-07      8.2550E-11
-----------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 56.38
Degree of consolidation        Us [%]  = 56.38
Settlement                     s  [cm] = 5.61
```

```
Degree of consolidation/ Settlement:
----------------------------------------------------

T [Years]      23.000      46.000      69.000

----------------------------------------------------
  Us  [%]       56.38       76.56       87.39
   s [cm]        5.61        7.62        8.70
----------------------------------------------------
```



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 887	**Project: A study on one-dimensional consolidation.... Lee (1992)**
File: Equivalent layer (SD)	**Date: 14-09-2017**
Page No.:	**Title: Consolidation of a Equivalent Layer for High Values of Tv (DD)**

5.5.9 Example 8: Consolidation of Multiple Soil Layers (10-layers)

5.5.9.1 Description of the problem:

Multiple soil layers (10-layers) are considered with *Cv* varies from 4.5×10^{-6} to 4.5×10^{-8} [m^2/sec]. For case of a system having too many layers with extreme differences in soil properties, the convergence of the solution may be not occurred without stability care. The program *GEO Tools* overcomes this problem automatically by choosing sub layers lead to the parameter of the coefficient of consolidation and thickness, $\mu_i = \frac{h_i / h_1}{\sqrt{C_{v1}/C_{vi}}}$ for all layers nearly equal to 1.

In this test example, it is sufficient to choose the number of sub layers for the 10 sub layers such that: 4, 2, 4, 13, 4, 2, 4, 13, 4, 2. To get the same accuracy by *FDM*, each layer must be subdivided into 20 sub-layers at least. Therefore, more sub-layers as indicated in the Table 5.12 are considered.

The below results of *LEM* show a good agreement with those of *FDM* of small sub layer thickness.

5.5.9.2 Data:

Initial pore water pressure	*uo*	[kN/m^2]	= 100
Total layer thickness	H_d	[m]	= 20
Depth increment in z-direction	*Di*	[m]	= 0.1

Table 5.12 Soil data

Layer No.	Layer Thickness *h* [m]	*LEM*		*FDM*		Coefficient of consolidation *Cv* [m^2/s]	Coefficient of permeability *K* [m/s]
		No. of sub layers	sub layer thickness [m]	No. of sub layers	sub layer thickness [m]		
1	2.0	20	0.10	20	0.10	4.756×10^{-7}	10^{-10}
2	2.0	7	0.29	20	0.10	4.756×10^{-6}	
3	2.0	20	0.10	20	0.10	4.756×10^{-7}	
4	2.0	64	0.03	20	0.10	4.756×10^{-8}	
5	2.0	20	0.10	20	0.10	4.756×10^{-7}	
6	2.0	7	0.29	20	0.10	4.756×10^{-6}	
7	2.0	20	0.10	20	0.10	4.756×10^{-7}	
8	2.0	64	0.03	20	0.10	4.756×10^{-8}	
9	2.0	20	0.10	20	0.10	4.756×10^{-7}	
10	2.0	7	0.29	20	0.10	4.756×10^{-6}	

5.5.9.3 Results:

Results of *LEM* are compared with those of *FDM* in Table 5.13 and Table 5.14, Figure 5.55 and Figure 5.56 at different times. One can see that results obtained from *LEM* are in a good agreement with those of *FDM* for both cases of double and single drainage.

Table 5.13 Comparison of results for pervious bottom boundary

Time factor/ Years	*LEM*		*FDM*	
	UP %	US %	UP %	US %
t=5 Years	57.18	46.18	58.42	47.71
t=10 Years	75.75	69.56	77.10	71.24
t=20 Years	92.21	90.23	93.04	91.27

Table 5.14 Comparison of results for impervious bottom boundary

Time factor/ Years	*LEM*		*FDM*	
	UP %	US %	UP %	US %
t=5 Years	30.21	21.94	30.73	22.55
t=10 Years	41.61	34.99	42.41	35.88
t=20 Years	58.36	53.73	59.40	54.89

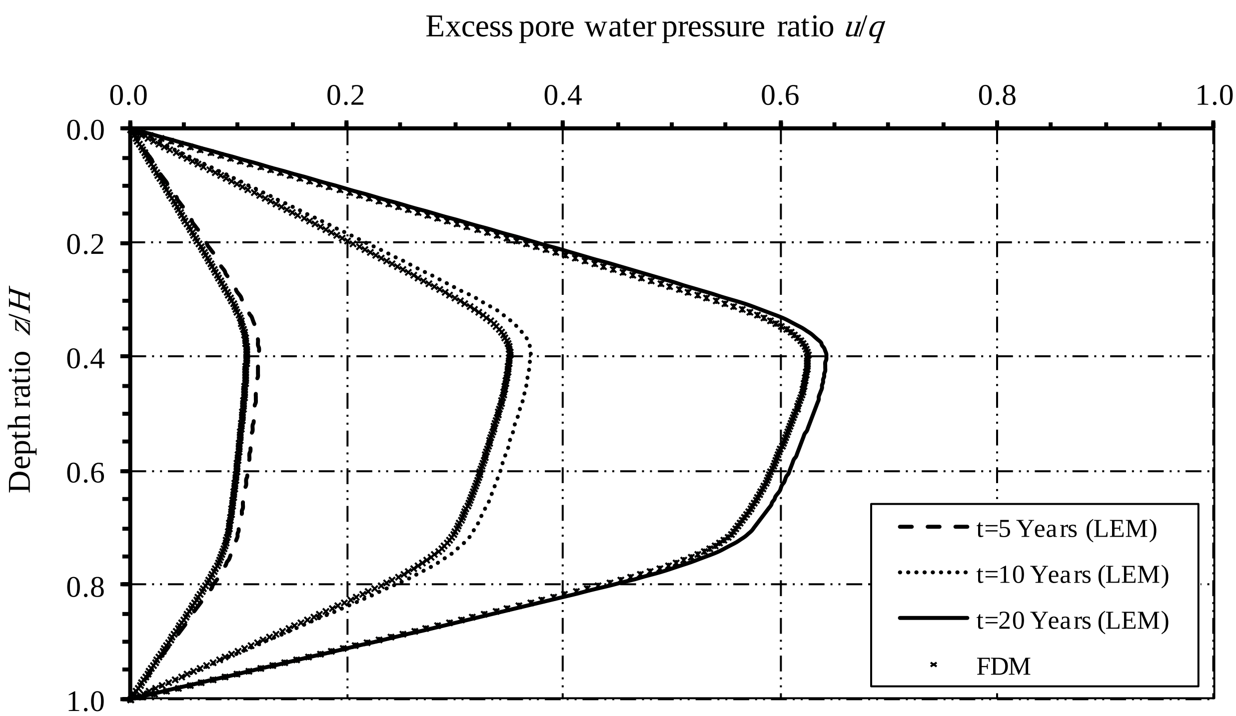


Figure 5.55 Excess pore water pressure ratio with depth ratio at different *t* (double drainage)

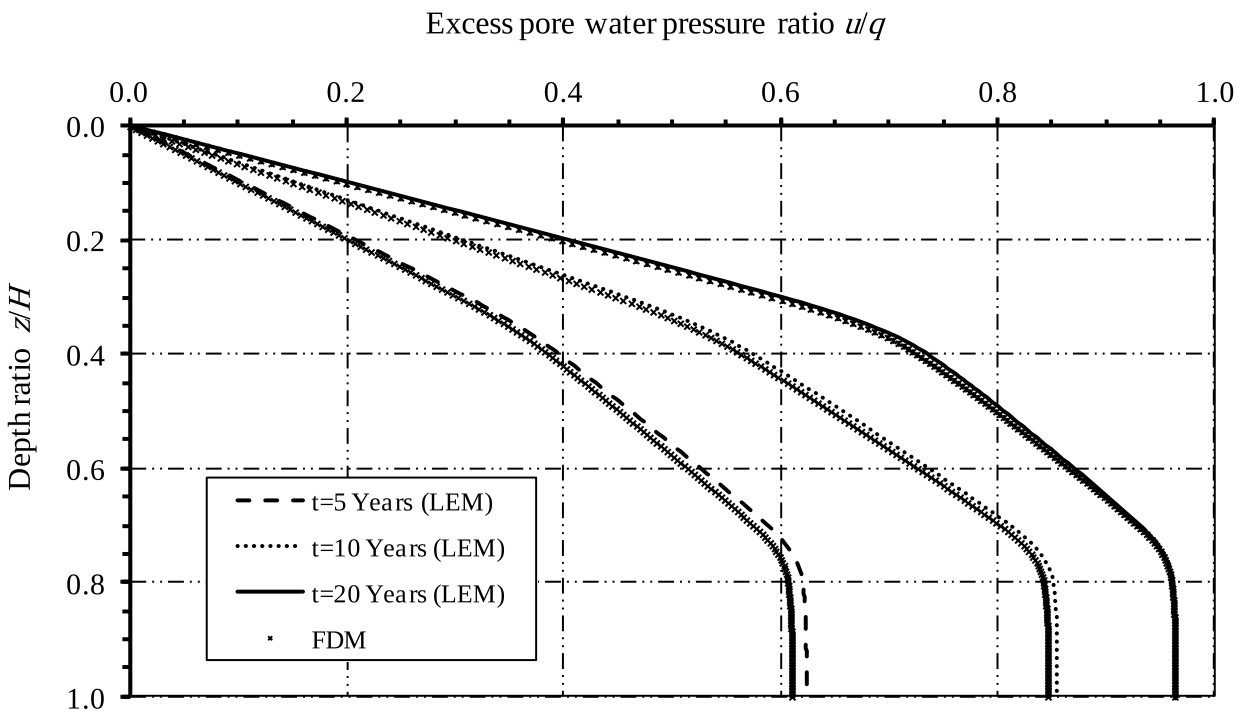


Figure 5.56 Excess pore water pressure ratio with depth ratio at different *t* (single drainage)

5.5.9.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation by *LEM* and *FDM* for single and double drainages are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Multiple soil layers (10-layers)-DD
Date: 15-09-2017
Project: Test of the number of sublayers
File: 10 Multiple layers(DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 20.00
Depth increment in z-direction       Di          [m]     = 0.10

Time:
Time of consolidation                Tr          [Years] = 5.000

Generation of times:
Start time                           To          [Years] = 5.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              5.000
  2             10.000
----------------------
```

Boring:

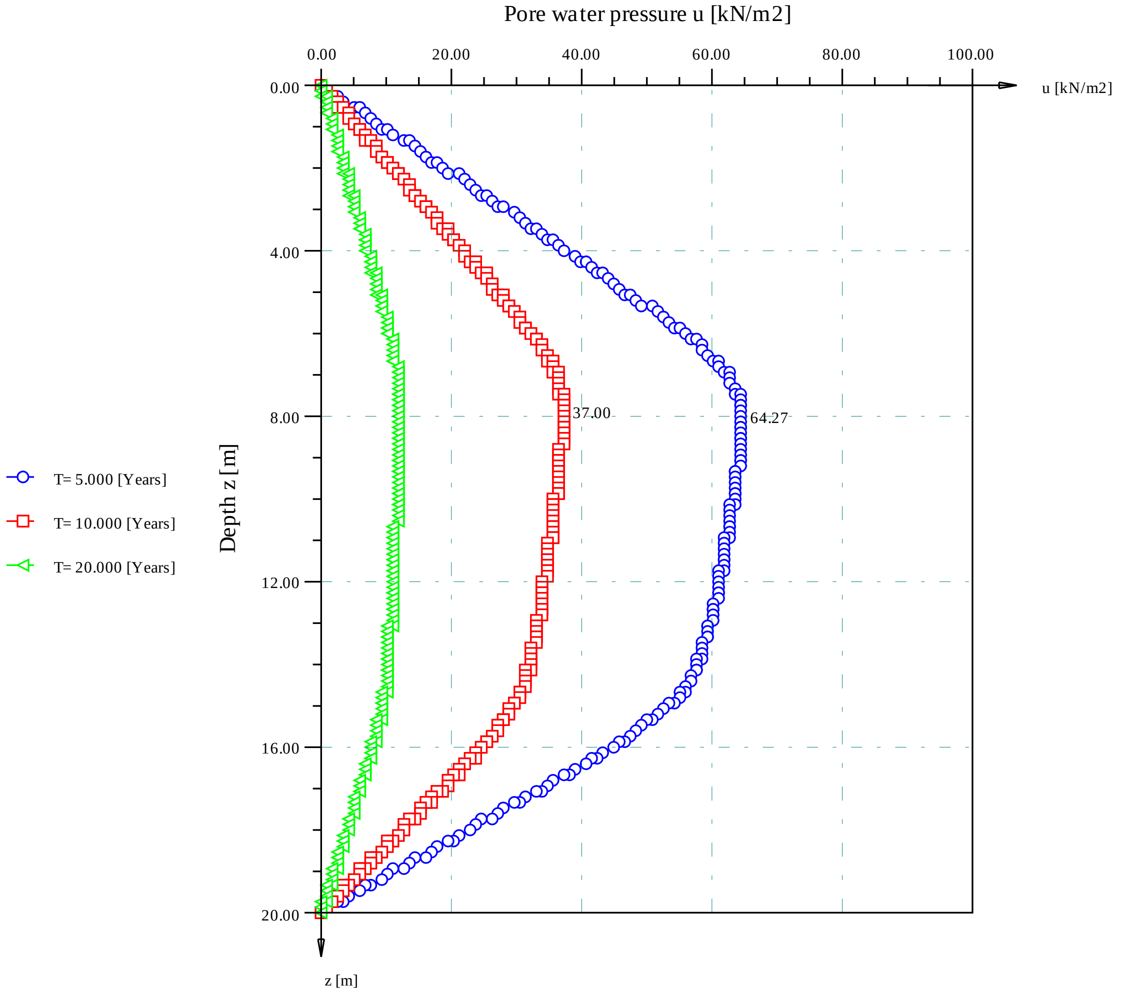
Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	2.00	20	4.7560E-07	1.0000E-10
2	2.00	7	4.7560E-06	1.0000E-10
3	2.00	20	4.7560E-07	1.0000E-10
4	2.00	64	4.7560E-08	1.0000E-10
5	2.00	20	4.7560E-07	1.0000E-10
6	2.00	7	4.7560E-06	1.0000E-10
7	2.00	20	4.7560E-07	1.0000E-10
8	2.00	64	4.7560E-08	1.0000E-10
9	2.00	20	4.7560E-07	1.0000E-10
10	2.00	7	4.7560E-06	1.0000E-10

Results:

```
Degree of consolidation        Up [%]  = 57.18
Degree of consolidation        Us [%]  = 46.18
Settlement                     s  [cm] = 5.01
```

Degree of consolidation/ Settlement:

T [Years]	5.000	10.000	20.000
Us [%]	46.18	69.56	90.23
s [cm]	5.01	7.54	9.79



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 844	**Project: Test of the number of sublayers**
File: 10 Multiple layers(DD)	**Date: 15-09-2017**
Page No.:	**Title: Multiple soil layers (10-layers)**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Multiple soil layers (10-layers)-DD
Date: 15-09-2017
Project: Test of the number of sublayers
File: 10 Multiple layers(DD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 20.00
Depth increment in z-direction       Di          [m]     = 0.10

Time:
Time of consolidation                Tr          [Years] = 5.000
Time increment                       dT          [Years] = 0.250

Generation of times:
Start time                           To          [Years] = 5.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              5.000
  2             10.000
----------------------
```

Boring:

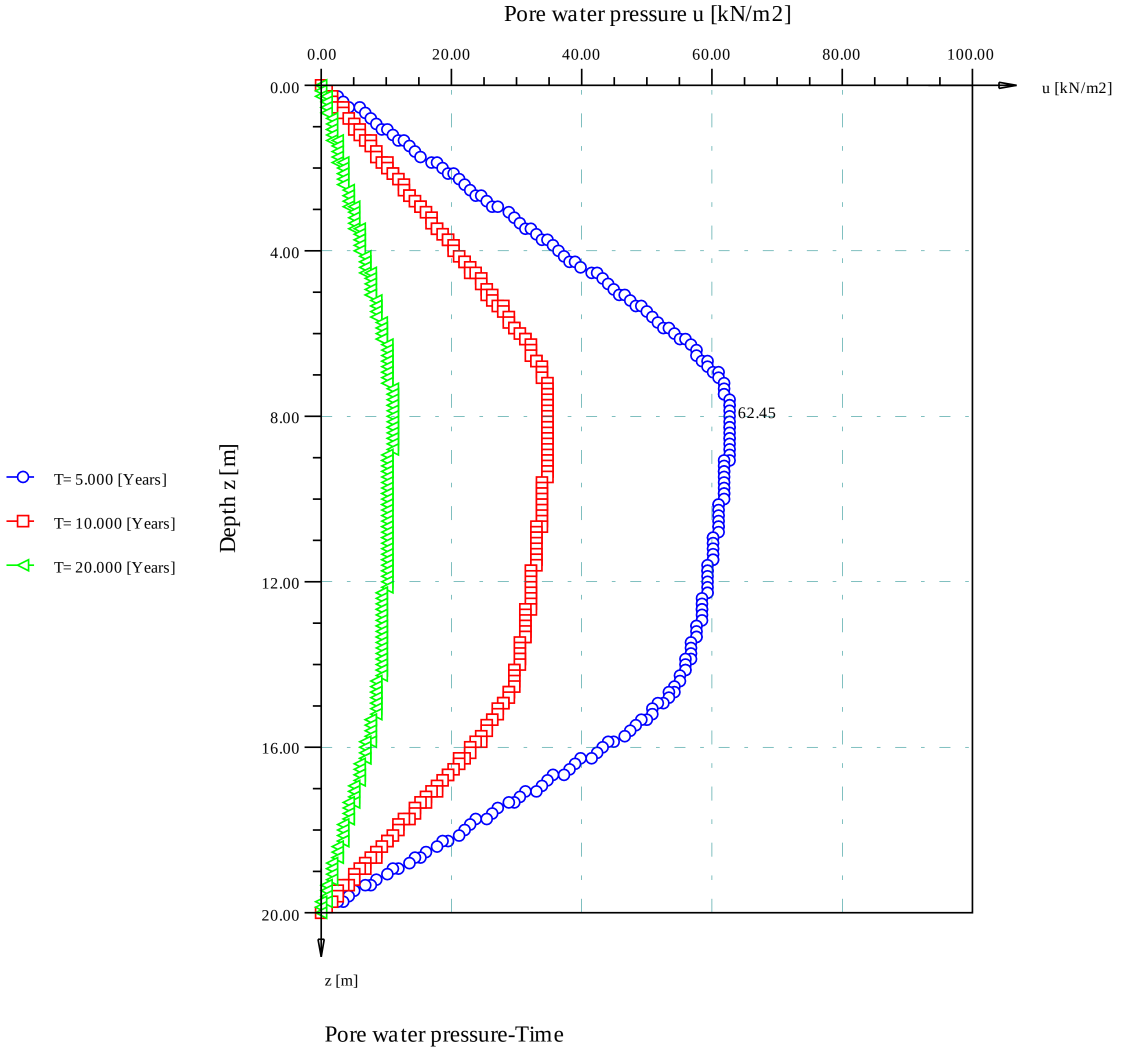
Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	2.00	20	4.7560E-07	1.0000E-10
2	2.00	20	4.7560E-06	1.0000E-10
3	2.00	20	4.7560E-07	1.0000E-10
4	2.00	20	4.7560E-08	1.0000E-10
5	2.00	20	4.7560E-07	1.0000E-10
6	2.00	20	4.7560E-06	1.0000E-10
7	2.00	20	4.7560E-07	1.0000E-10
8	2.00	20	4.7560E-08	1.0000E-10
9	2.00	20	4.7560E-07	1.0000E-10
10	2.00	20	4.7560E-06	1.0000E-10

Results:

```
Degree of consolidation        Up [%]  = 58.42
Degree of consolidation        Us [%]  = 47.71
Settlement                     s  [cm] = 5.17
```

Degree of consolidation/ Settlement:

T [Years]	5.000	10.000	20.000
Us [%]	47.71	71.24	91.27
s [cm]	5.17	7.73	9.90



GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 844	**Project: Test of the number of sublayers**
File: 10 Multiple layers(DD)	**Date: 15-09-2017**
Page No.:	**Title: Multiple soil layers (10-layers)**

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Multiple soil layers (10-layers)-SD
Date: 15-09-2017
Project: Test of the number of sublayers
File: 10 Multiple layers (SD)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 20.00
Depth increment in z-direction       Di          [m]     = 0.10

Time:
Time of consolidation                Tr          [Years] = 5.000

Generation of times:
Start time                           To          [Years] = 5.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              5.000
  2             10.000
----------------------
```

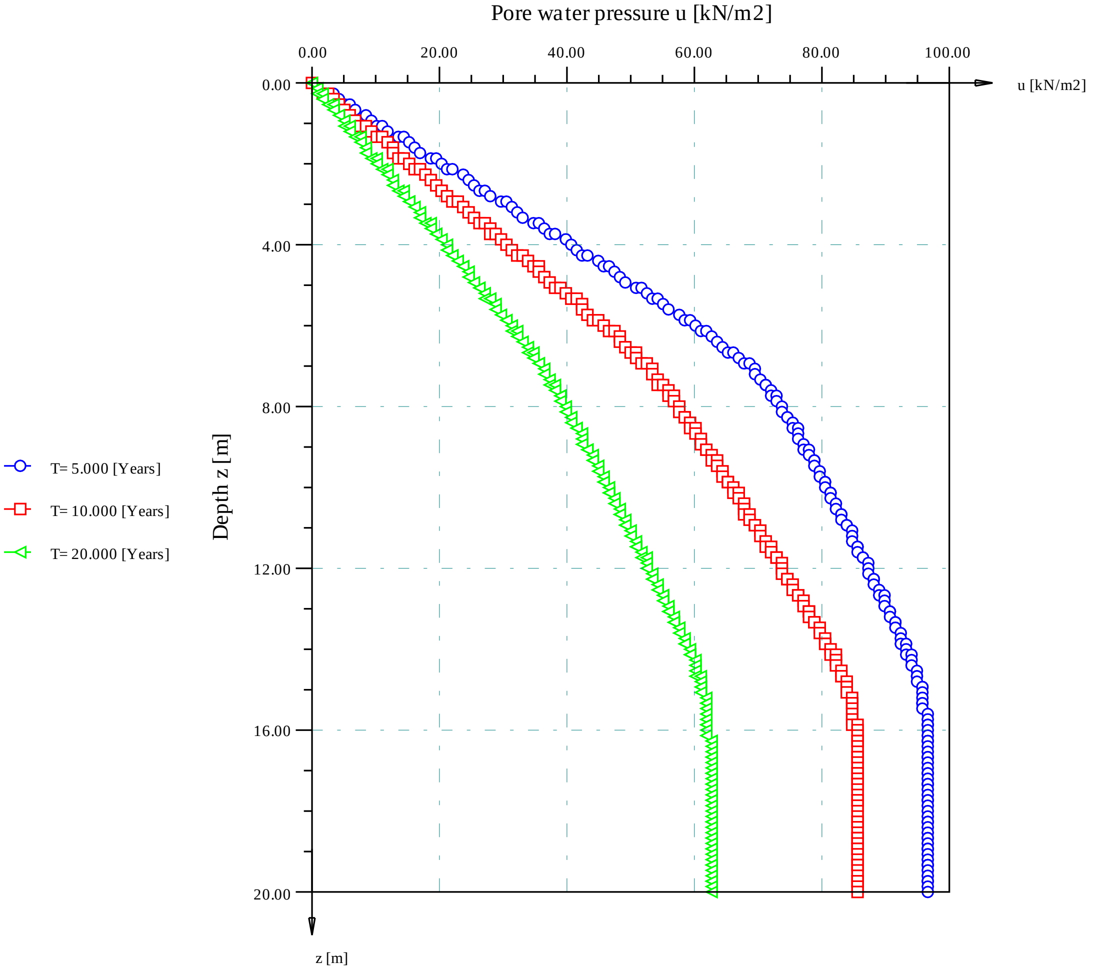
Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	2.00	20	4.7560E-07	1.0000E-10
2	2.00	7	4.7560E-06	1.0000E-10
3	2.00	20	4.7560E-07	1.0000E-10
4	2.00	64	4.7560E-08	1.0000E-10
5	2.00	20	4.7560E-07	1.0000E-10
6	2.00	7	4.7560E-06	1.0000E-10
7	2.00	20	4.7560E-07	1.0000E-10
8	2.00	64	4.7560E-08	1.0000E-10
9	2.00	20	4.7560E-07	1.0000E-10
10	2.00	7	4.7560E-06	1.0000E-10

Results:
Degree of consolidation Up [%] = 30.20
Degree of consolidation Us [%] = 21.94
Settlement s [cm] = 2.38

Degree of consolidation/ Settlement:

T [Years]	5.000	10.000	20.000
Us [%]	21.94	34.99	53.73
s [cm]	2.38	3.79	5.83



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 846	**Project: Test of the number of sublayers**
File: 10 Multiple layers (SD)	**Date: 15-09-2017**
Page No.:	**Title: Multiple soil layers (10-layers)-SD**

```
************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Multiple soil layers (10-layers)-SD
Date: 15-09-2017
Project: Test of the number of sublayers
File: 10 Multiple layers (SD)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Finite Difference Method (FDM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo         [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 20.00
Depth increment in z-direction       Di         [m]     = 0.10

Time:
Time of consolidation                Tr         [Years] = 5.000
Time increment                       dT         [Years] = 0.250

Generation of times:
Start time                           To         [Years] = 5.000

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]            [Years]
----------------------
  1              5.000
  2             10.000
----------------------
```

Boring:

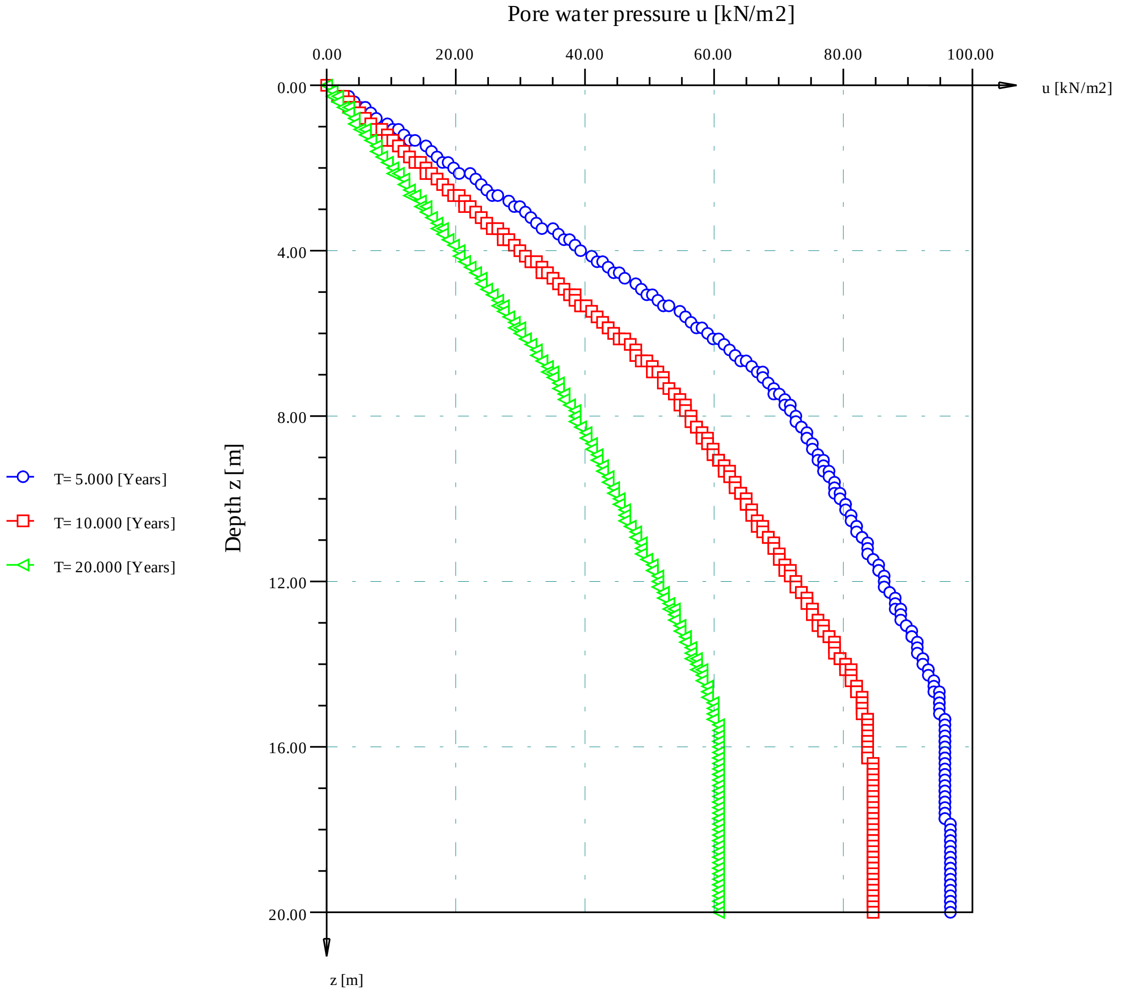
Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	2.00	20	4.7560E-07	1.0000E-10
2	2.00	20	4.7560E-06	1.0000E-10
3	2.00	20	4.7560E-07	1.0000E-10
4	2.00	20	4.7560E-08	1.0000E-10
5	2.00	20	4.7560E-07	1.0000E-10
6	2.00	20	4.7560E-06	1.0000E-10
7	2.00	20	4.7560E-07	1.0000E-10
8	2.00	20	4.7560E-08	1.0000E-10
9	2.00	20	4.7560E-07	1.0000E-10
10	2.00	20	4.7560E-06	1.0000E-10

Results:

Degree of consolidation Up [%] = 30.73
Degree of consolidation Us [%] = 22.55
Settlement s [cm] = 2.45

Degree of consolidation/ Settlement:

T [Years]	5.000	10.000	20.000
Us [%]	22.55	35.88	54.89
s [cm]	2.45	3.89	5.95



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7	
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 856	**Project: Test of the number of sublayers**
File: 10 Multiple layers (SD)	**Date: 15-09-2017**
Page No.:	**Title: Multiple soil layers (10-layers)-SD**

5.6 Example to Verify Consolidation Rate under Linear Loading

5.6.1 Example 9: Consolidation of a Linear Loading due to Uniform Initial Stress

5.6.1.1 Description of the problem:

Conte and Troncone (2006) presented an example for the consolidation due to a load that first increase linearly from zero up to a load intensity q_c and then remains constant with time using an analytical solution depending on Fourier series. To verify *LEM* for time-dependent settlement of a linear loading, the excess pore water pressure ratio and the time factor calculated analytically by *Conte and Troncone* (2006) are compared with those obtained by *LEM*.

5.6.1.2 Data:

Initial pore water pressure	*uo*	[kN/m^2]	= 1.0
Total layer thickness	H_d	[m]	= 10
Depth increment in z-direction	*Di*	[m]	= 1.0
Coefficient of consolidation	C_v	[m^2/ sec]	= 3×10^{-15}
Coefficient of permeability	k_v	[m/ sec]	= 2.94×10^{-10}
Time of construction	t_c	[Days]	= 13

The periodic function plotted in Figure 5.57 is used to represent the load that first increase linearly from zero up to a load intensity q_c and then remains constant with time. In the figure t_b is the time defines the end of loading stage, and t_b is a time chosen so that the consolidation is expected to be practically completed in the time interval (0, t_b). Times of consolidation are chosen to be t_b = 3.846, 7.692, 15.385, 26.923 and 34.615 [days], which give time factors of T_v = 0.1, 0.2, 0.4 , 0.7 and 0.9 [-], respectively.

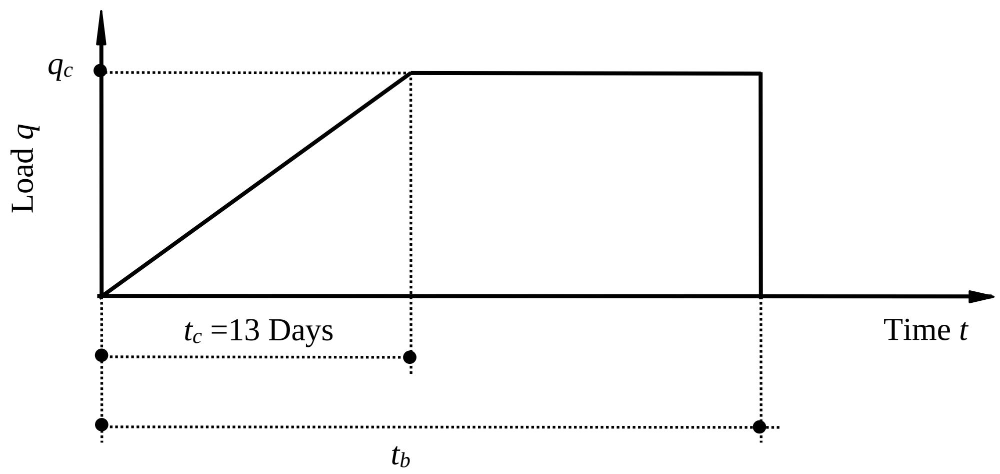


Figure 5.57 Loading scheme

5.6.1.3 Results:

Results of *LEM* show a good agreement with those of *Conte and Troncone* (2006) at different values of time factors as shown in Figure 5.58.

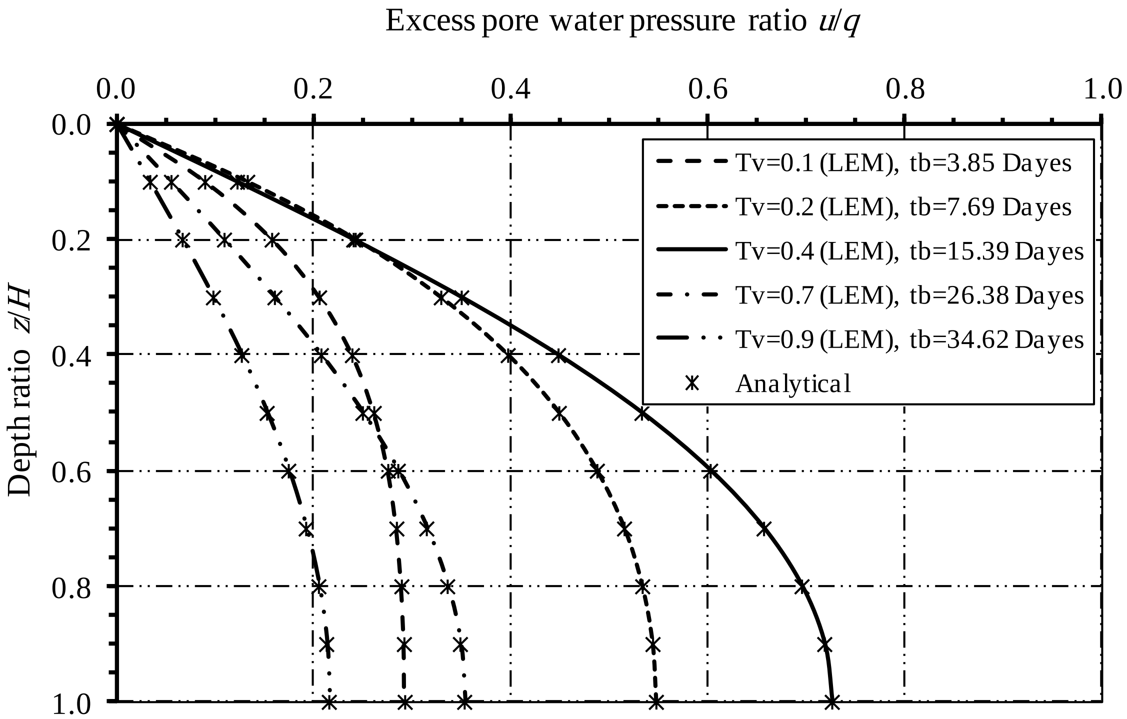


Figure 5.58 Excess pore water pressure ratio with depth ratio at different *Tv* (double drainage)

5.6.1.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: One-dimensional consolidation under..., Conte and Troncone (2006)
Date: 15-09-2017
Project: Conte and Troncone (2006)
File: Conte and Troncone (2006)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Linear loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure       uo          [kN/m2] = 1.00
Overburden pressure                Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                    Hb          [m]     = 10.00
Depth increment in z-direction     Di          [m]     = 1.00

Time:
Time of consolidation              Tr          [Days]  = 100.000
Time of construction               Tc          [Days]  = 13.000

Generation of times:
Start time                         To          [Days]  = 3.850

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]             [Days]
----------------------
  1              3.850
  2              7.690
  3             11.538
  4              7.690
----------------------

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      10.00         10      3.0000E-05      2.9400E-10
------------------------------------------------------------
```

Results:

Degree of consolidation	Up [%]	= 99.79
Degree of consolidation	Us [%]	= 99.79
Settlement	s [cm]	= 0.00

Initial and Final pore water pressures with depth:

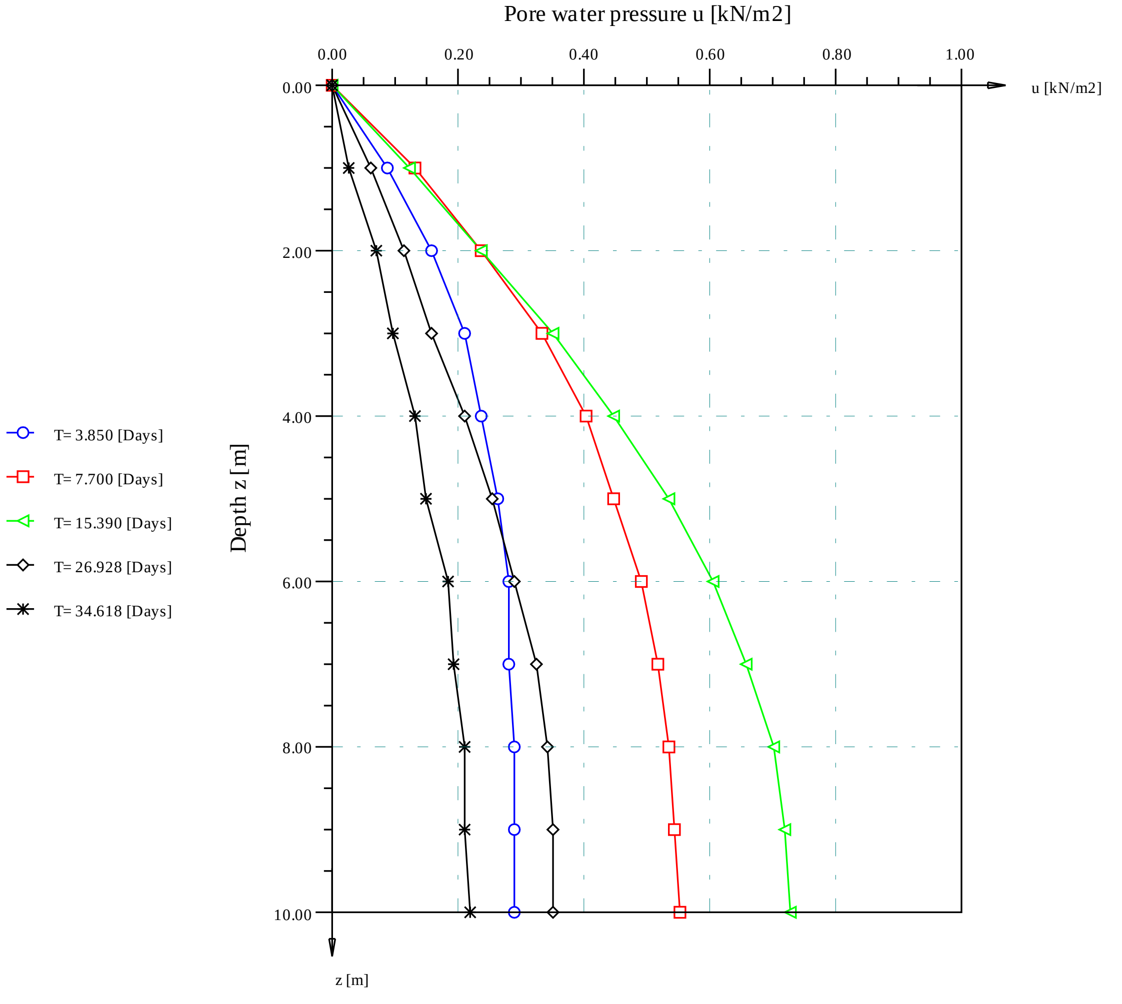
No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	1.00	0.00
1	1.00	1.00	0.00
2	2.00	1.00	0.00
3	3.00	1.00	0.00
4	4.00	1.00	0.00
5	5.00	1.00	0.00
6	6.00	1.00	0.00
7	7.00	1.00	0.00
8	8.00	1.00	0.00
9	9.00	1.00	0.00
10	10.00	1.00	0.00

Pore water pressure U [kN/m2]:

T [Days] \ z [m]	3.850	7.700	15.390	26.928	34.618
0.00	0.00	0.00	0.00	0.00	0.00
1.00	0.09	0.13	0.12	0.06	0.03
2.00	0.16	0.24	0.24	0.11	0.07
3.00	0.21	0.33	0.35	0.16	0.10
4.00	0.24	0.40	0.45	0.21	0.13
5.00	0.26	0.45	0.53	0.25	0.15
6.00	0.28	0.49	0.60	0.29	0.18
7.00	0.28	0.52	0.66	0.32	0.19
8.00	0.29	0.53	0.70	0.34	0.21
9.00	0.29	0.54	0.72	0.35	0.21
10.00	0.29	0.55	0.73	0.35	0.22

Degree of consolidation/ Settlement:

T [Days]	3.850	7.700	15.390	26.928	34.618
Us [%]	77.51	60.79	52.56	77.46	86.22
s [cm]	0.00	0.00	0.00	0.00	0.00



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 9	**Project: One-dimensional consolidation under..., Conte and Troncone (2006)**
File: Linear loading	**Date: 15-09-2017**
Page No.:	**Title: Consolidation of a Linear Loading**

5.6.2 Example 10: Consolidation of a Linear Loading due to Variable Initial Stress

5.6.2.1 Description of the problem

Liu and *Griffiths* (2015) presented a general analytical solution for obtaining the excess pore water pressure in a consolidating layer due to depth and time-dependent changes of total stress. The solution of *Liu* and *Griffiths* (2015) was verified with three special cases, one of them were chosen to verify the *LEM* in *GEO Tools*. The chosen verification example was originally considered by *Zhu* and *Yin* (1998). *Zhu* and *Yin* (1998) considered a linearly increasing time-dependent "ramp" load with linearly varying total stress distribution with depth. The ramp reached a maximum $t_m = t_c$ and remained constant thereafter.

Zhu and *Yin* (1998) assumed single-drained condition for a consolidation clay layer of total thickness H = 10 [m] and coefficient of consolidation $C_v = 3 \times 10^{-6}$ [m²/ sec.]. The variation of the stress with depth was assumed to be linear with maximum values at the top and bottom of the layer given by σ_t = 150 [kN/ m²] and σ_b = 50 [kN/ m²] respectively as shown in Figure 5.59. Assume the coefficient of consolidation $k_v = 1 \times 10^{-08}$ [m²/ sec].

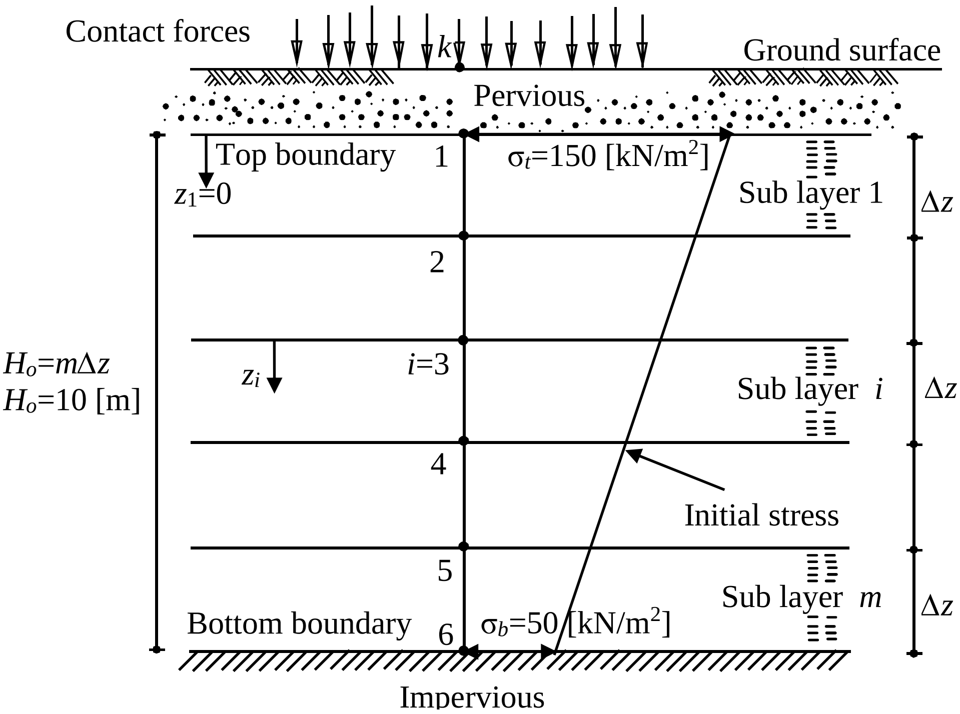


Figure 5.59 Initial excess pore water pressure on the clay layer

The periodic function plotted in Figure 5.60 is used to represent the load that first increase linearly from zero up to a load intensity q_c until the time t_c =0.5 years, and then remains constant with time. In the figure t_c is the time defines the end of loading stage, and t_b is a time chosen so that the consolidation is expected to be practically completed in the time interval (0, t_b). Time of consolidation is chosen to be t_b = 4 years.

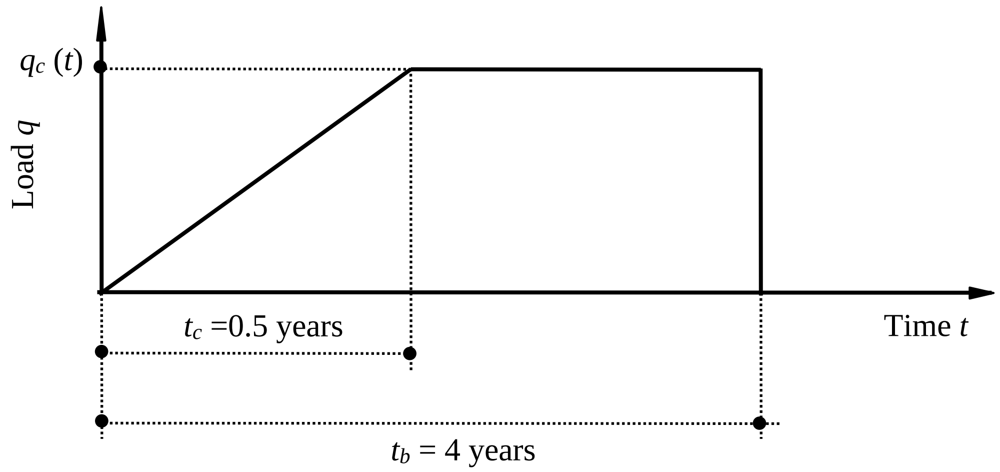


Figure 5.60 Loading scheme

5.6.2.2 Analysis of the problem

The clay layer is divided into 5 layers each of 2 [m] thick, and the time is divided into 40 intervals, each of 0.1 year.

5.6.2.3 Results and discussions

As shown in Figure 5.61, which show the variation of excess pore water pressure with time at the layer base, the results of the *LEM* are identical with those of *Liu* and *Griffiths* (2015).

5.6.2.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* are presented on the next pages.

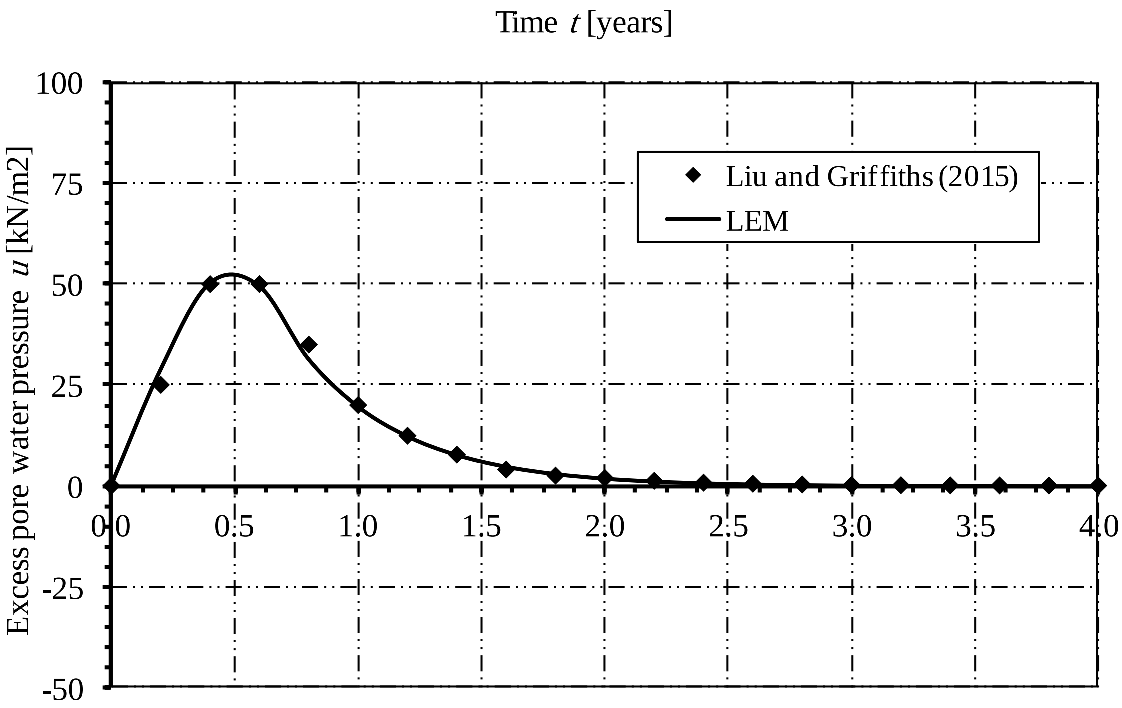


Figure 5.61 Variation of excess pore water pressure *u* with time *t* at the layer base

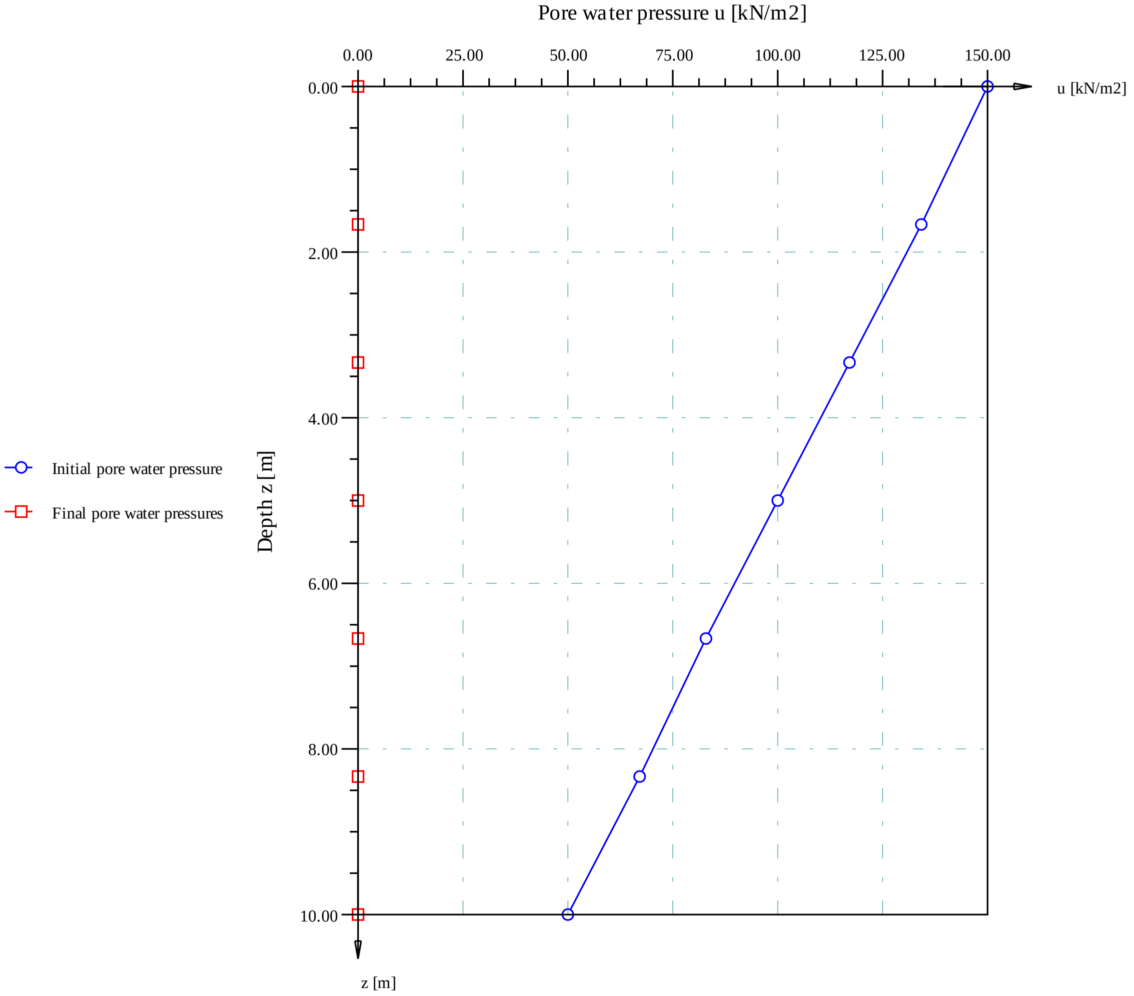
```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of soil under depth-dependent ramp load
Date: 29-09-2017
Project: Zhu and Yin (1998)
File: Zhu and Yin (1998)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Linear loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Pore water pressure is defined by the user
Overburden pressure                   Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                       Hb         [m]     = 10.00
Depth increment in z-direction        Di         [m]     = 2.00

Time:
Time of consolidation                 Tr         [Years] = 4.00
Time of construction                  Tc         [Years] = 0.50

Generation of times:
Start time                            To         [Years] = 0.00
No. of time intervals                 Nt         [-]     = 40
Time interval                         Ti         [Years] = 0.10

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      10.00          5      3.0000E-06      1.0000E-08
------------------------------------------------------------

Results:
Degree of consolidation        Up [%]  = 99.99
Degree of consolidation        Us [%]  = 99.99
Settlement                     s  [cm] = 33.98
```

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	150.00	0.00
1	1.67	133.33	0.00
2	3.33	116.67	0.01
3	5.00	100.00	0.01
4	6.67	83.33	0.02
5	8.33	66.67	0.02
6	10.00	50.00	0.02

Initial and Final pore water pressures with depth:

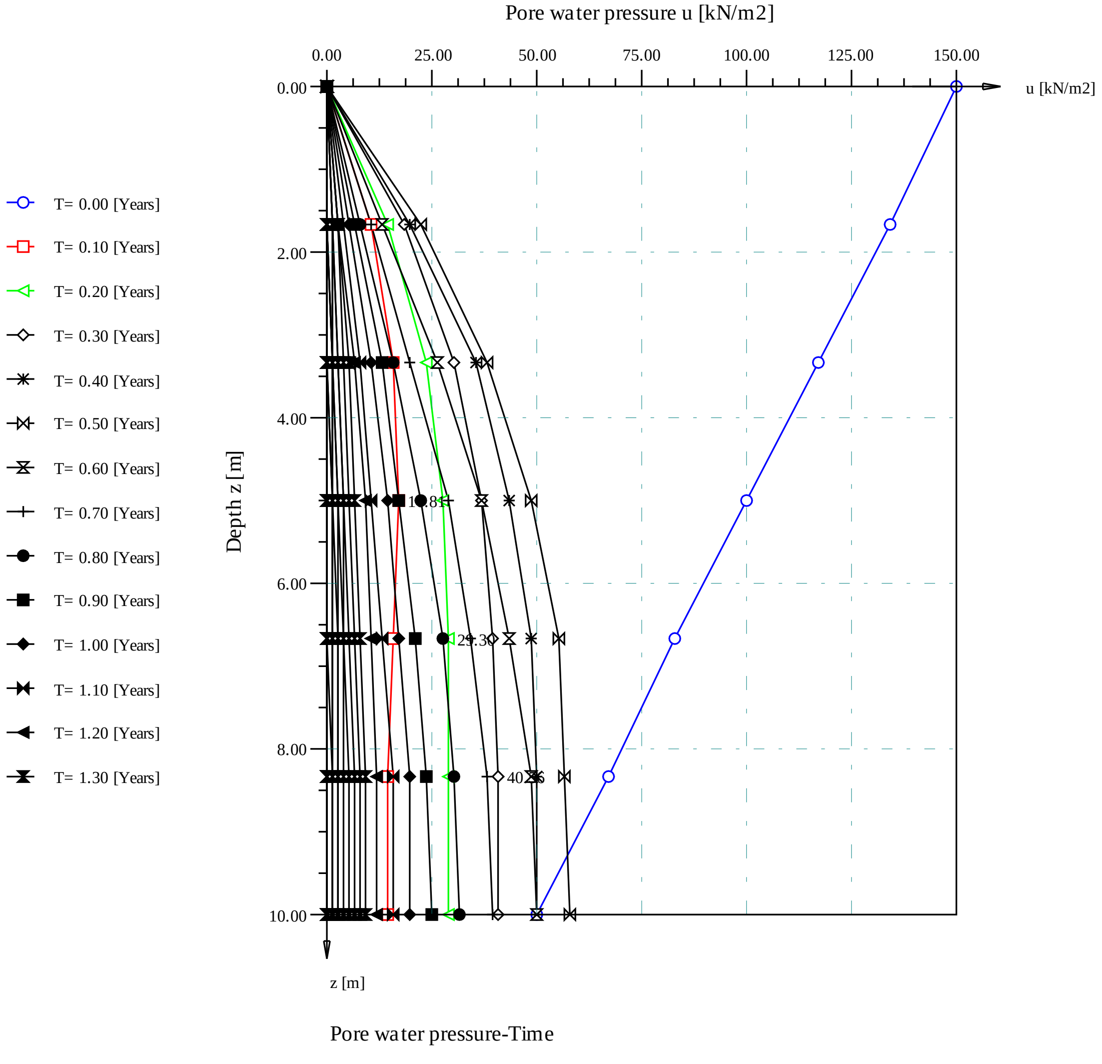
No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
1	0.00	150.00	0.00
2	10.00	50.00	0.02



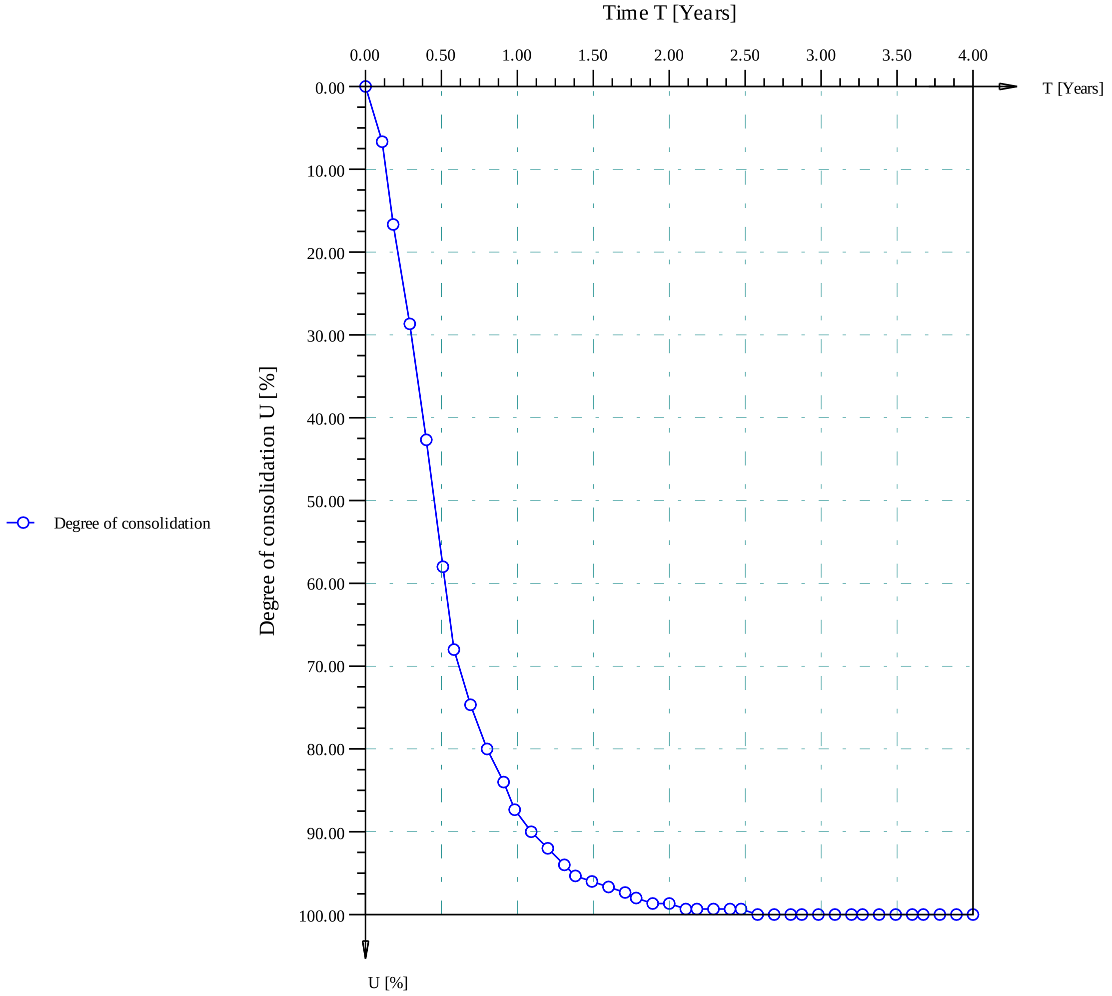
Pore water pressure

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 1313	**Project: Zhu and Yin (1998)**
File: Zhu and Yin (1998)	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of soil under depth-dependent ramp load**



GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1313	**Project: Zhu and Yin (1998)**
File: Zhu and Yin (1998)	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of soil under depth-dependent ramp load**



Degree of consolidation-Time

<table>
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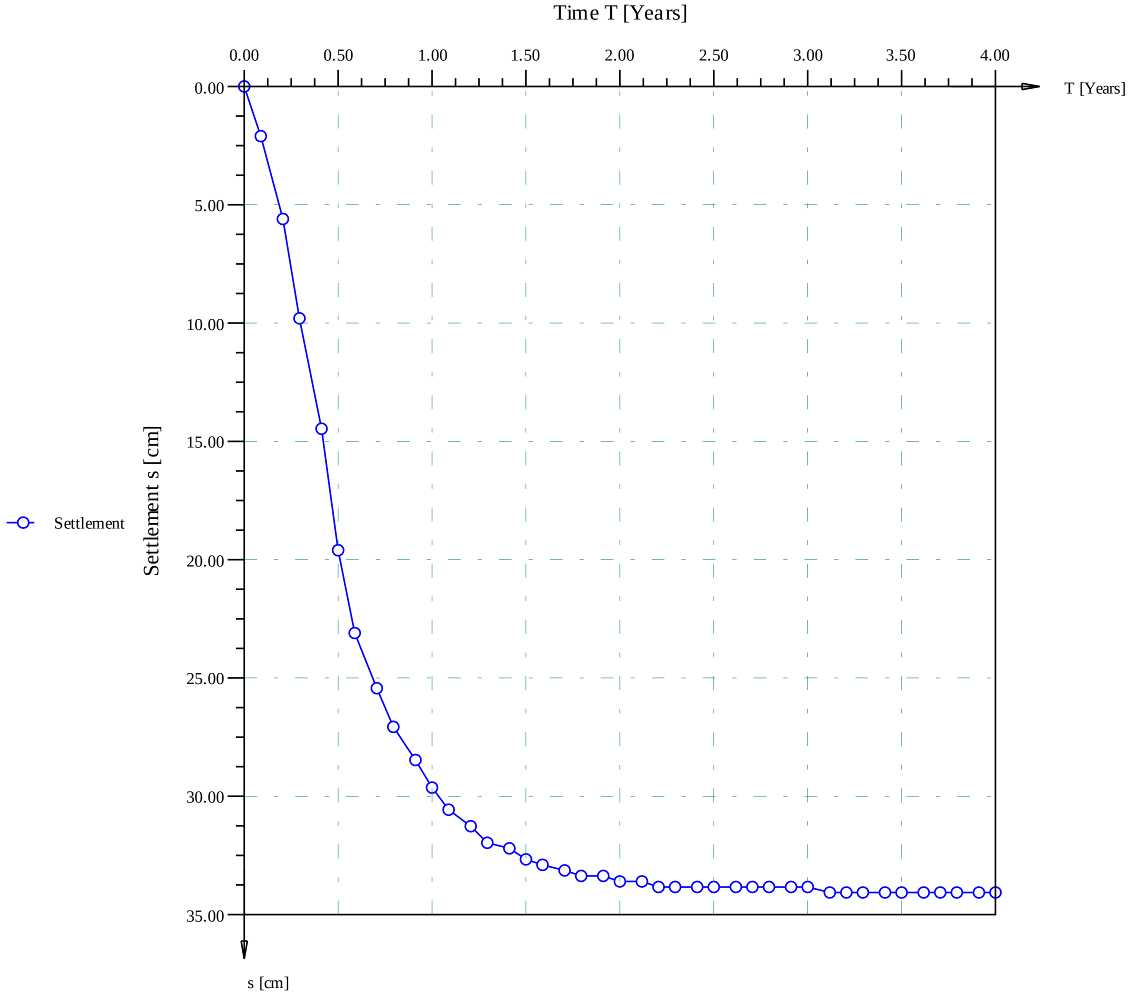
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323</td></tr>
<tr><td>Scale: 37
File: Zhu and Yin (1998)
Page No.:</td><td>Project: Zhu and Yin (1998)
Date: 29-09-2017
Title: Consolidation of soil under depth-dependent ramp load</td></tr>
</table>



Settlement-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 29	**Project: Zhu and Yin (1998)**
File: Zhu and Yin (1998)	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of soil under depth-dependent ramp load**

5.6.3 Example 11: Consolidation of a Constant and a Linear Loading

5.6.3.1 Description of the problem

Hamza and *Maksimovic* (1988) presented the time-settlement curve for a saturated clay layer, which is loaded linearly over a wide area at the surface, using a graphical construction depending on the instantaneous curve. To verify *LEM* for determining time-settlement curve of either a constant or a linear loading, the time-settlement curves obtained graphically by *Hamza* and *Maksimovic* (1988) are compared with those obtained by *LEM*.

A surface of a ground is loaded over a wide area by q = 80 [kN/ m^2] as shown in Figure 5.62. The saturated clay layer is H = 3 [m] thick and has a double-drained condition. The coefficient of consolidation of the clay is $C_v = 8 \times 10^{-8}$ [m^2/ sec.], while that of permeability is $k_v = 2.7 \times 10^{-10}$ [m^2/ sec]. It is required to plot the time-settlement curve up to 95% of consolidation of the clay for the two following cases:

a) The load is applied instantaneously,
b) The load is applied linearly over a time period of 250 days, and then remains constant.

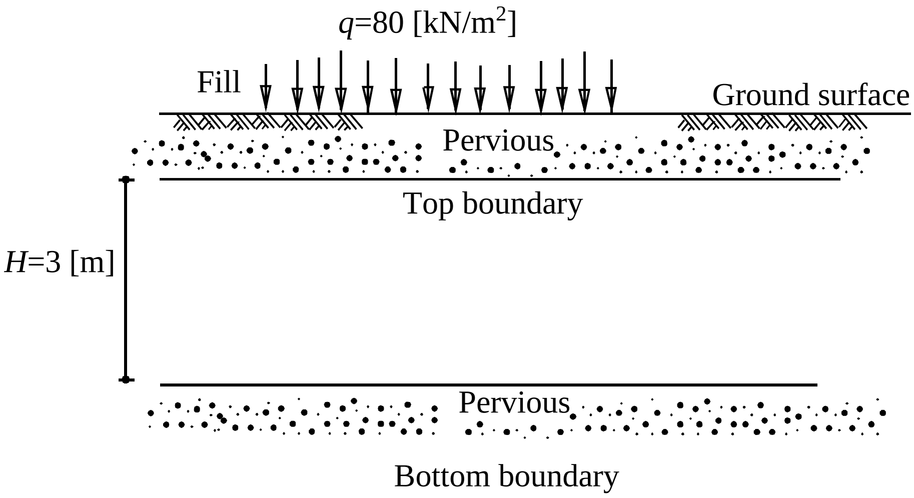


Figure 5.62 Saturated clay layer with loading on the surface

5.6.3.2 Analysis of the problem by Hamza and Maksimovic (1988)

a) The load is applied instantaneously

Modulus of compressibility E_s:

$$E_s = \frac{C_v \gamma_w}{k_v}$$

$$E_s = \frac{8 \times 10^{-8} \times 9.81}{2.7 \times 10^{-10}} = 2906 \text{ [kN/m}^2\text{]}$$

Final settlement S_c:

$$S_c = \frac{1}{E_s} \Delta\sigma \, H$$

$$S_c = \frac{1}{2906} 80 \times 3 = 0.083 \text{ [m]}$$

$$S_c = 8.3 \text{ [cm]}$$

Settlement of time t:

$$S(t) = S_c \, U(t)$$

Relation between time t and time factor T_v is given by:

$$t = \frac{T_v \, H_d^2}{C_v}$$

$$t = \frac{T_v \, 1.5^2}{8 \times 10^{-8} \times 3600 \times 24} = 325.52 \, T_v \text{ [days]}$$

Degree of consolidation $U(t)$ %, time factor T_v, time t [days] and settlement $S(t)$ [cm] are presented in Table 5.15 and presented in diagrams of Figure 5.63 to Figure 5.65.

Table 5.15 Time-settlement values

U%	10	20	30	40	50	60	70	80	90	95
T_v	0.008	0.031	0.071	0.126	0.197	0.287	0.403	0.567	0.848	1.127
t [days]	2.6	10.09	23.11	41.02	64.13	93.42	131.2	184.6	276.0	366.9
$S(t)$ [cm]	0.83	1.65	2.48	3.30	4.13	4.96	5.78	6.61	7.43	7.85

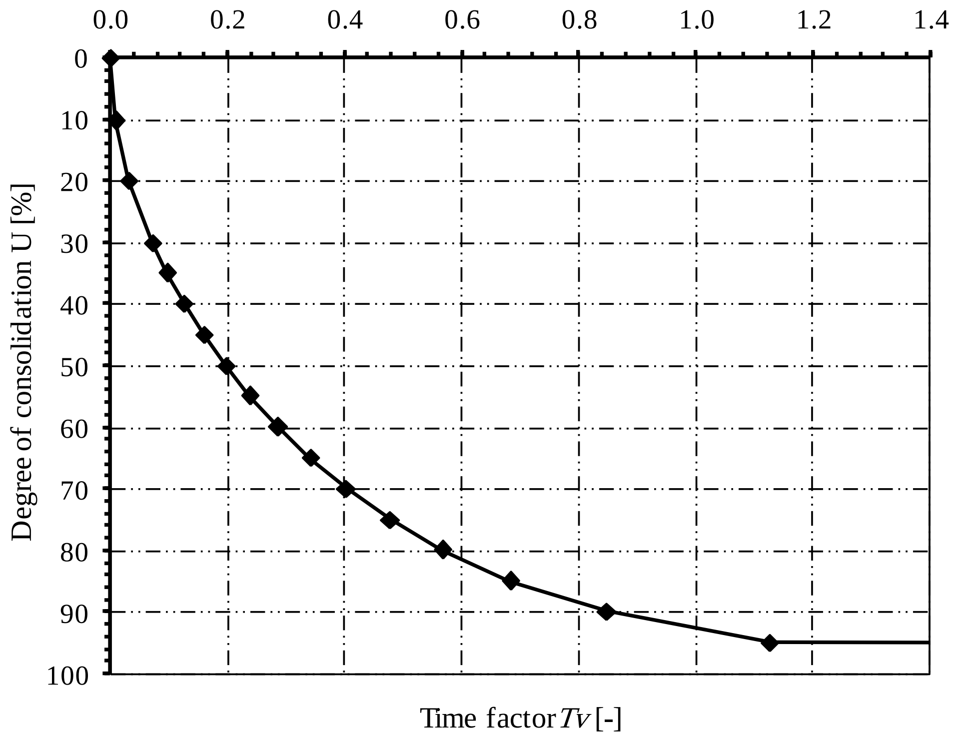


Figure 5.63 Degree of consolidation U % with time factor T_v [-]

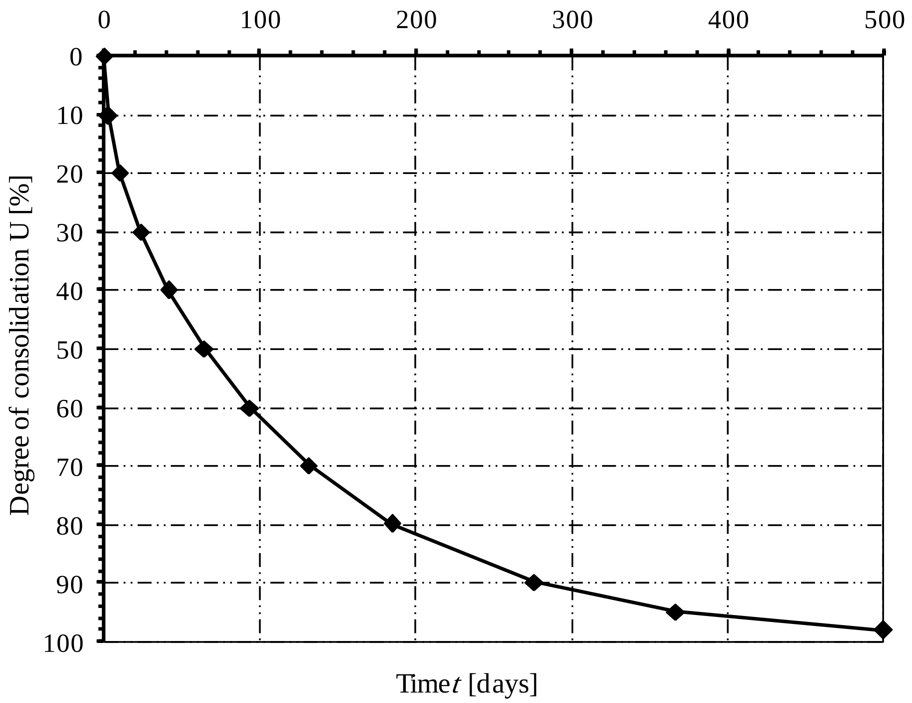


Figure 5.64 Degree of consolidation with time for instantaneous loading

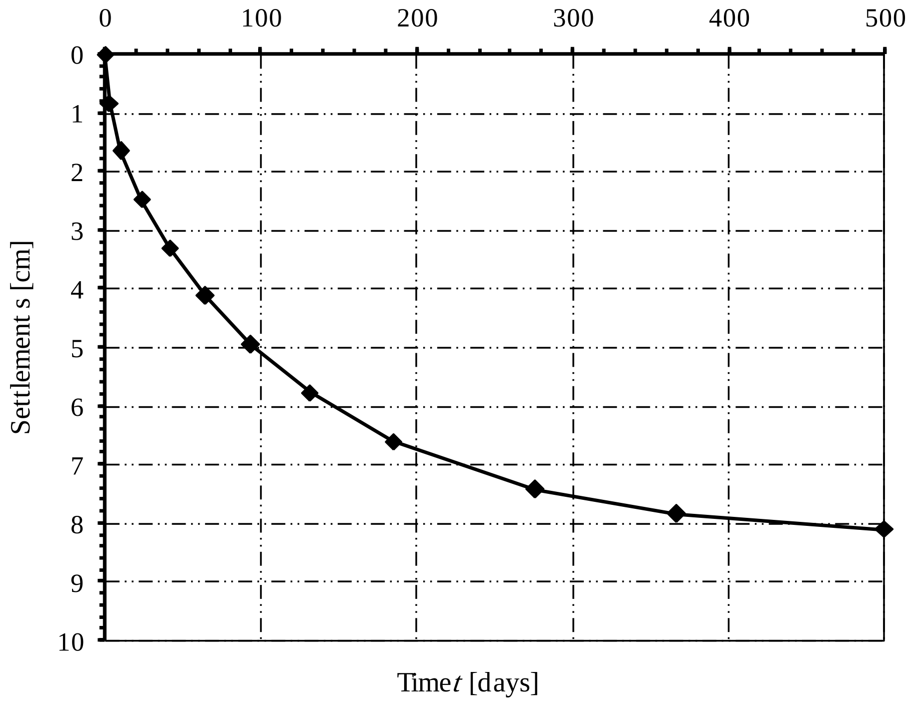


Figure 5.65 Time-settlement curve for instantaneous loading

b) The load is applied linearly over a time period of 250 days, and then remain constant.

The periodic function plotted in Figure 5.66 is used to represent the load that first increase linearly from zero up to a load intensity q_c =80 [kN/m^2] until the time t_c =250 days, and then remains constant with time. In the figure t_c is the time defines the end of loading stage, and t_b is a time chosen so that the consolidation is expected to be practically completed in the time interval (0, t_b).

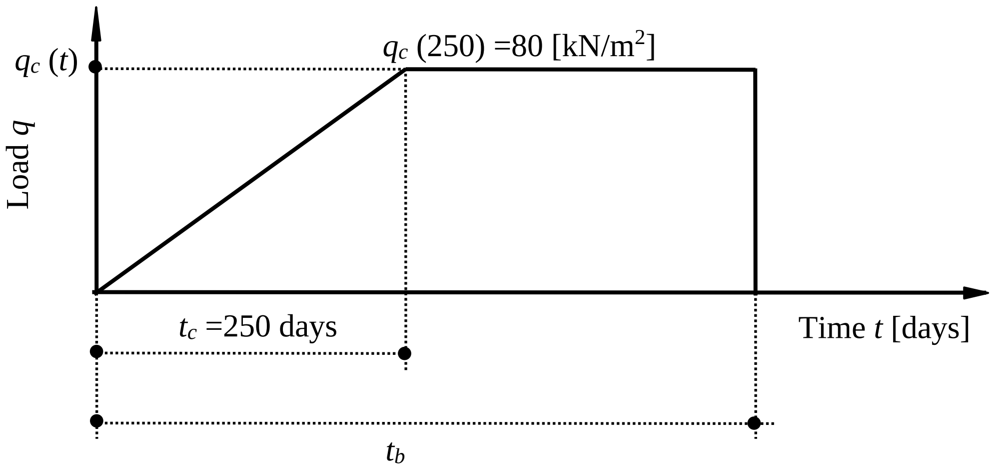


Figure 5.66 Loading scheme

The time-settlement curve for the linear increase of load is obtained from the instantaneous curve using graphical construction demonstrated in Figure 5.67. In Figure 5.67 t_G is the duration of the effective construction period. Figure 5.68 and Figure 5.69 show the degree of consolidation and settlement with time those obtained by *GEO Tools*.

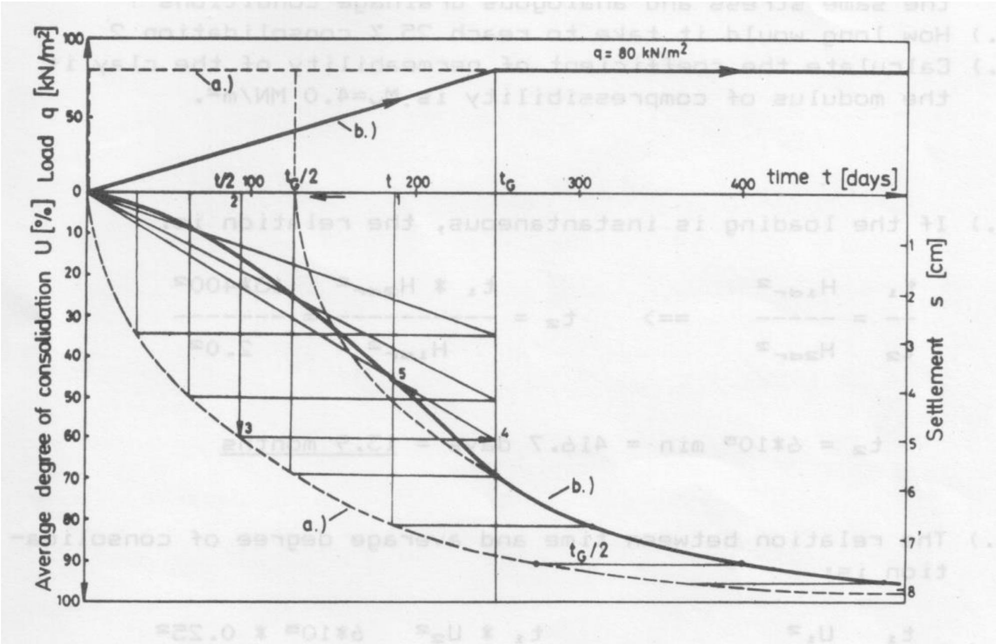


Figure 5.67 Time-settlement curve for linear loading (*Hamza* and *Maksimovic* (1988))

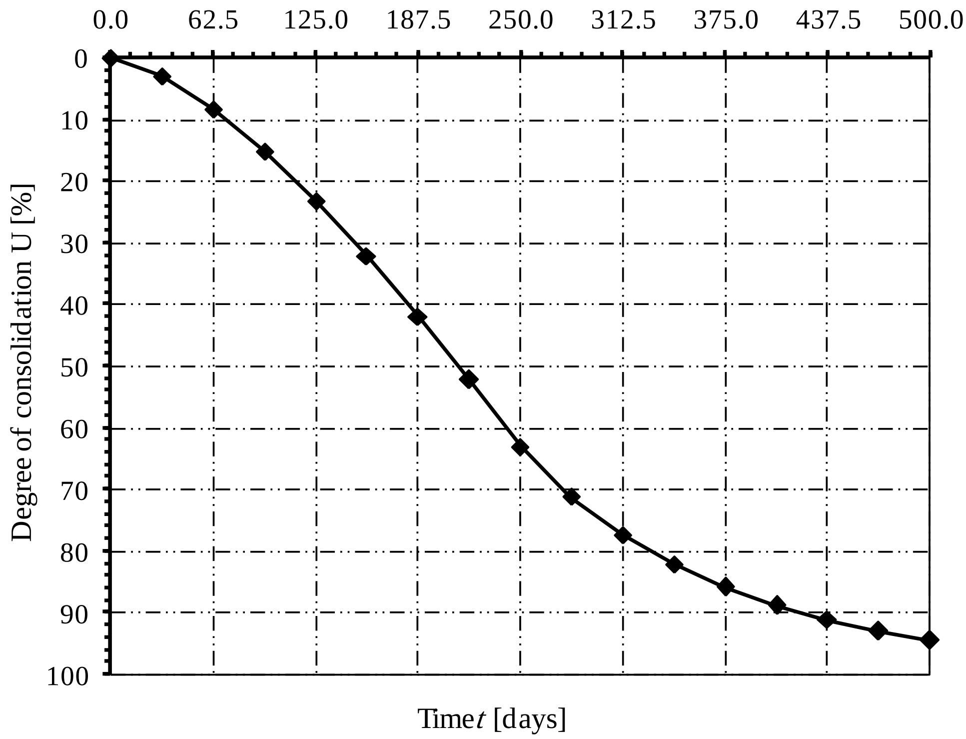


Figure 5.68 Degree of consolidation with time for linear loading (*GEO Tools*)

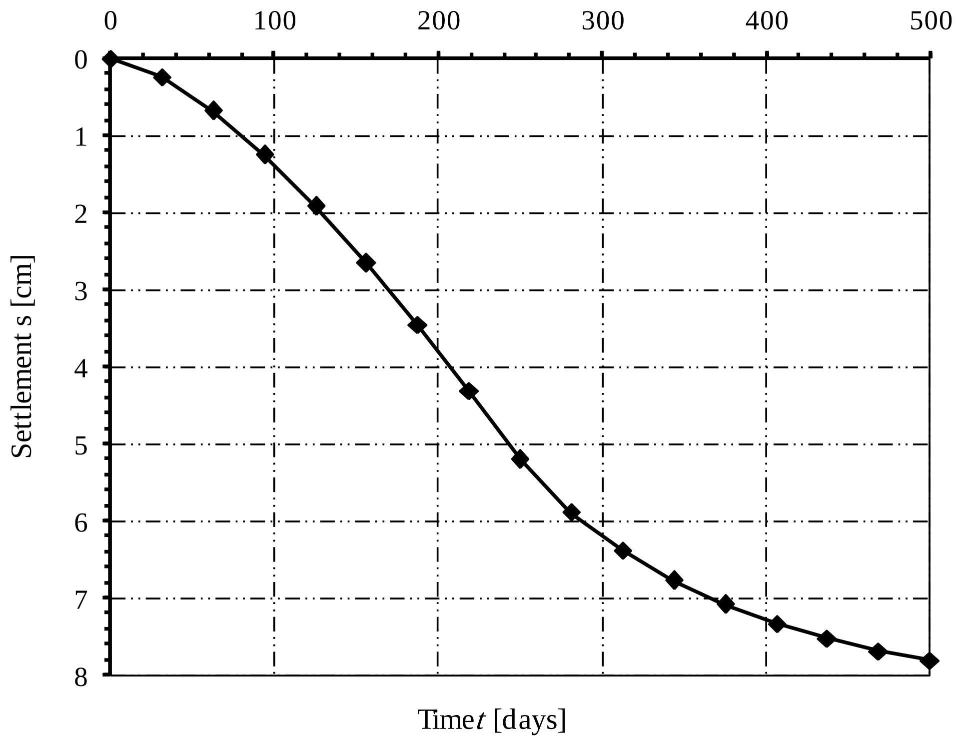


Figure 5.69 Time-settlement curve for linear loading (*GEO Tools*)

5.6.3.3 Analysis of the problem

The clay layer is divided into 38 layers each of 0.08 [m] thick, and the time is divided into 10 variable intervals as given in the original example.

5.6.3.4 Results and discussions

As shown from results presented by *Hamza* and *Maksimovic* (1988) and those obtained by *GEO Tools*, that the results of the *LEM* are identical with those of *Hamza* and *Maksimovic* (1988).

5.6.3.5 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 11
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Collection of solved problems in soil mechanics
Date: 31/03/2018
Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Constant loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Pore water pressure is defined by the user
Overburden pressure                  Po=Gamma*z [kN/m2] = 0

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 3.00
Depth increment in z-direction       Di         [m]     = 0.08

Time:
Time of consolidation                Tr         [Days]  = 500.00

Generation of times:
Start time                           To         [Days]  = 0.00

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]             [Days]
----------------------
  1               2.60
  2               7.49
  3              13.02
  4              17.91
  5              23.11
  6              29.29
  7              37.78
  8              53.40
  9              91.40
 10              90.90
 11             133.10
----------------------
```

Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	3.00	38	8.0000E-08	2.7000E-10

Results:

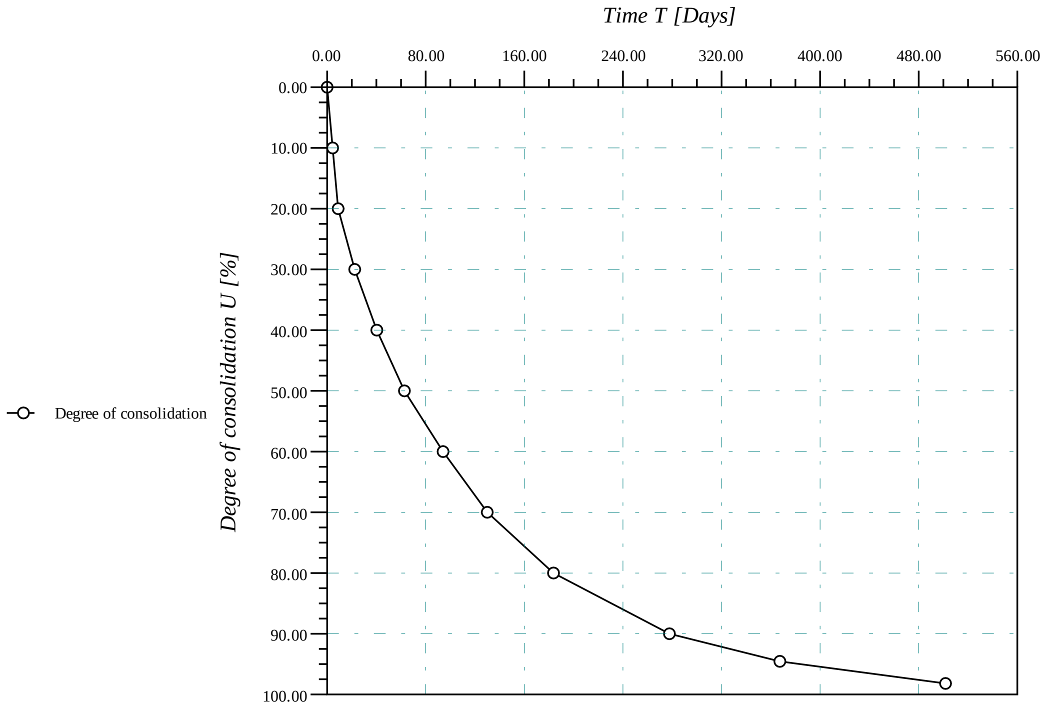
Degree of consolidation Up [%] = 98
Degree of consolidation Us [%] = 98
Settlement s [cm] = 8.11

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
1	0.00	80	0
2	3.00	80	0

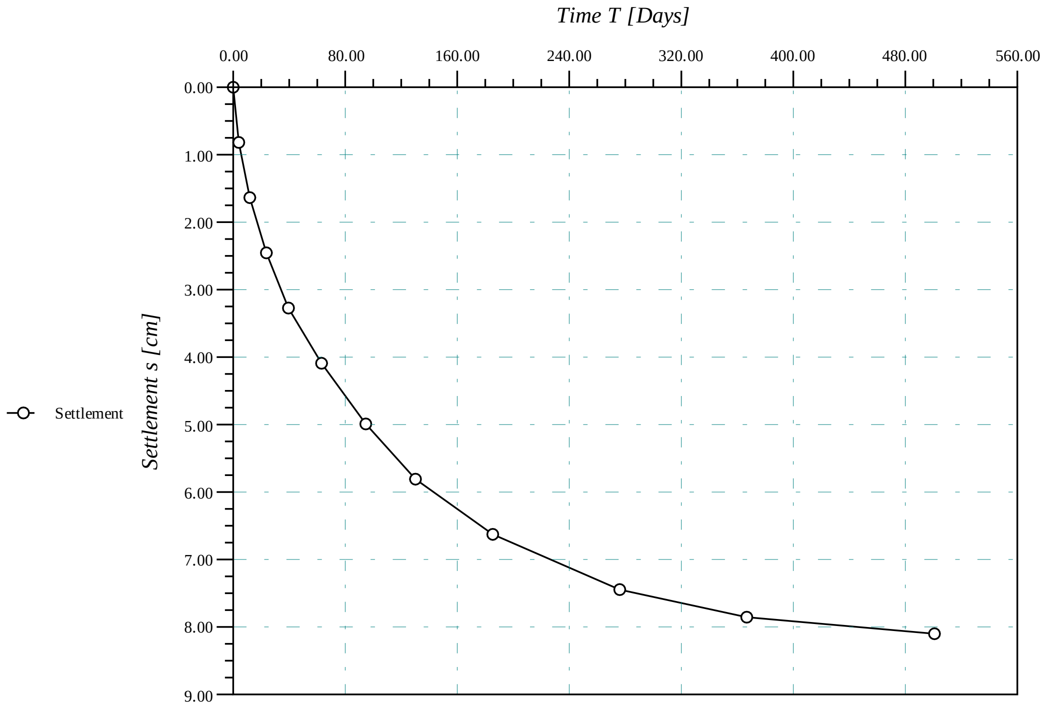
Degree of consolidation/ Settlement:

T [Days]	0.00	2.60	10.09	23.11	41.02	64.13	93.42	131.20	184.60	276.00	366.90	500.00
Us [%]	0	10	20	30	40	50	60	70	80	90	95	98
s [cm]	0.00	0.84	1.65	2.49	3.31	4.13	4.96	5.78	6.61	7.43	7.84	8.11



Degree of consolidation-Time

GEOTEC Software Inc.	
PO Box 14001 Richmond Road PO, Calgary AB, Canada T3E 7Y7	
Scale: 4480	Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)	Date: 31/03/2018
Page No.:	Title: Collection of solved problems in soil mechanics



Settlement-Time

GEOTEC Software Inc. PO Box 14001 Richmond Road PO, Calgary AB, Canada T3E 7Y7	
Scale: 3947	Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)	Date: 31/03/2018
Page No.:	Title: Collection of solved problems in soil mechanics

```
************************************************************
                        GEO Tools
                         Version 11
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
************************************************************
Title: Collection of solved problems in soil mechanics
Date: 31/03/2018
Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)

------------------------------------------------------------
Degree of consolidation
------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Linear loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Pore water pressure is defined by the user
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.0

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 3.00
Depth increment in z-direction       Di         [m]     = 0.08

Time:
Time of consolidation                Tr         [Days]  = 500.00
Time of construction                 Tc         [Days]  = 250.00

Generation of times:
Start time                           To         [Days]  = 0.00

Time intervals:
----------------------
No.      Time interval
  I                 Dt
[-]             [Days]
----------------------
  1               2.60
  2               7.49
  3              13.02
  4              17.91
  5              23.11
  6              29.29
  7              37.78
  8              53.40
  9              91.40
 10              90.90
 11             133.10
----------------------
```

Boring:

Layer No. I [-]	Layer thickness h [m]	No. of sublayers Nsl [-]	Coefficient of consolidation Cv [m2/s]	Coefficient of permeability k [m/s]
1	3.00	38	8.0000E-08	2.7000E-10

Results:

Degree of consolidation Up [%] = 94.54
Degree of consolidation Us [%] = 94.54

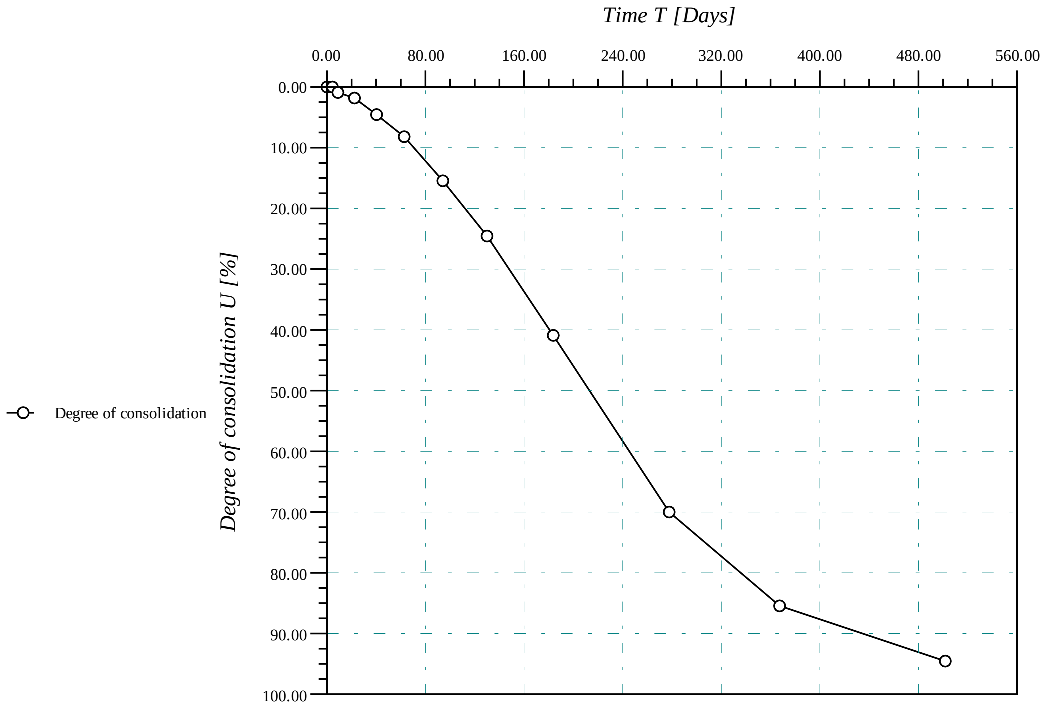
Settlement s [cm] = 7.81

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
1	0.00	80.0	0.0
2	3.00	80.0	0.0

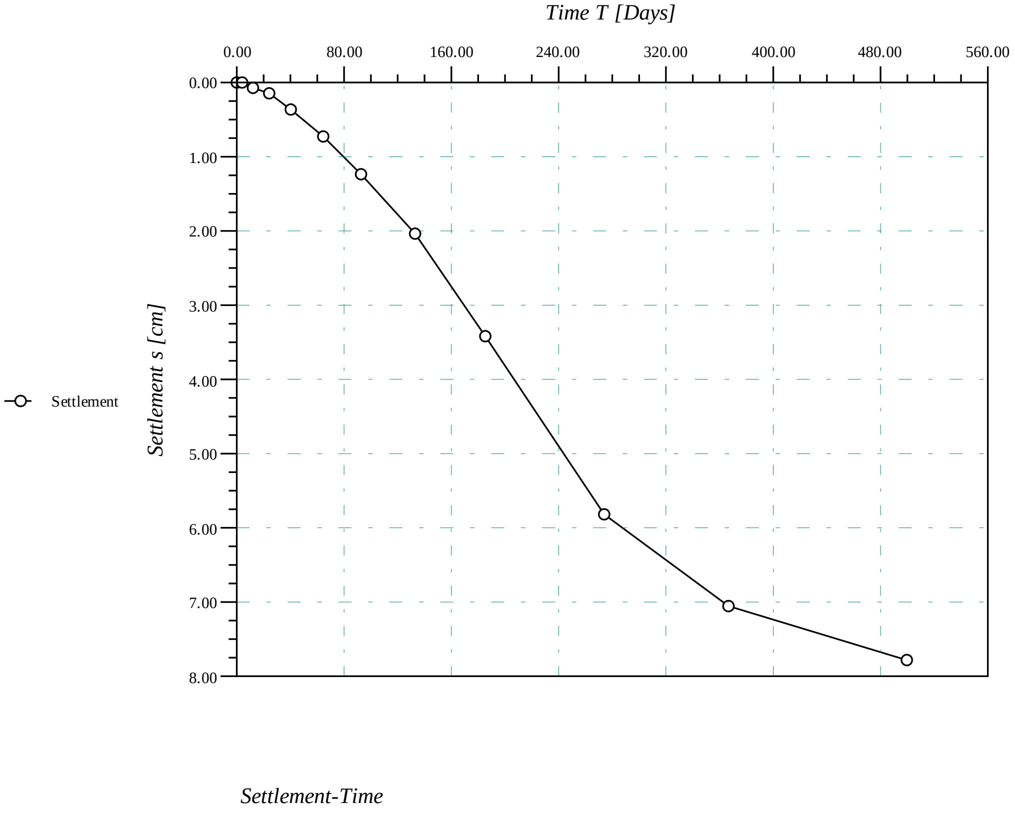
Degree of consolidation/ Settlement:

T [Days]	0.00	2.60	10.09	23.11	41.02	64.13	93.42	131.20	184.60	276.00	366.90	500.00
Us [%]	0.00	0.07	0.54	1.86	4.39	8.58	15.06	24.93	41.03	70.08	85.02	94.54
s [cm]	0.00	0.01	0.04	0.15	0.36	0.71	1.24	2.06	3.39	5.79	7.02	7.81



Degree of consolidation-Time

GEOTEC Software Inc. PO Box 14001 Richmond Road PO, Calgary AB, Canada T3E 7Y7	
Scale: 4480	Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)	Date: 31/03/2018
Page No.:	Title: Collection of solved problems in soil mechanics



GEOTEC Software Inc.

PO Box 14001 Richmond Road PO, Calgary AB, Canada T3E 7Y7

Scale: 4032	Project: Hamza and Maksimovic (1988)
File: Hamza and Maksimovic (1988)	Date: 31/03/2018
Page No.:	Title: Collection of solved problems in soil mechanics

5.7 Examples to Verify Consolidation Rate under Cyclic Loading

El Gendy, O. (2016) had carried out a numerical modification on the semi-analytical solution of *Toufig and Ouria* (2009) to be applicable for multi-layered soil subjected to any variable stress along the depth of the soil using *LEM*. Some of verification examples for cyclic loading on multi-layered soil carried out by him are presented in the next paragraphs.

5.7.1 Example 12: Consolidation under Rectangular Cyclic Loading

5.7.1.1 Description of the problem:

Toufigh and Ouria (2009) presented an example for the consolidation under a rectangular cyclic loading using a semi-analytical solution considering the effect of the change of the consolidation coefficient of the soil layer. To verify *LEM* in *GEO Tools* for the consolidation under a rectangular cyclic loading, the degree of consolidation *U* under a rectangular cyclic loading calculated by *Toufigh and Ouria* (2009) is compared with that obtained by *LEM*.

5.7.1.2 Data:

Initial pore water pressure	u_o	[kN/cm^2]	= 100
Total layer thickness	H_d	[cm]	= 1.8
Depth increment in z-direction	D_i	[cm]	= 0.18
Coefficient of consolidation	C_v	[cm^2/ min]	= 0.0029
Period of time	t_p	[min]	= 30
Time increment	dt	[min]	= 0.18
Loading/Reloading consolidation ratio $\frac{C_v(NC)}{C_v(OC)}$	β	[-]	= 0.095

Impervious bottom boundary

5.7.1.3 Analysis of the problem

For the analysis by *GEO Tools*, it is convenient to convert the unit system of the time period to days and the dimension to meters. For the same time factor, the coefficient of consolidation can be obtained from:

$$T_v = \frac{c_v[\text{cm}^2/\text{min}]\times t_p[\text{min}]}{H_d^2[\text{cm}]} = \frac{C_v[\text{m}^2/\text{day}]\times t_p[\text{day}]}{H_d^2[\text{m}]}$$

$$T_v = \frac{0.0029\ [\text{cm}^2/\text{min}]\times 30\ [\text{min}]}{1.8^2[\text{cm}]} = \frac{C_v[\text{m}^2/\text{day}]\times 30\ [\text{day}]}{1.8^2[\text{m}]}$$

$$C_v = 0.0029\ [\text{m}^2/\text{day}] = 3.35\times 10^{-8}\ [\text{m}^2/\text{sec}]$$

Then the equivalent example data with the new unit system will be:

Initial pore water pressure	u_o	[kN/m^2]	= 100
Total layer thickness	H_d	[m]	= 1.8
Depth increment in z-direction	D_i	[m]	= 0.18
Coefficient of consolidation	C_v	[m^2/ sec]	= 3.3565×10^{-8}

Coefficient of permeability	k_v	[m/ sec]	= 1×10^{-9}
Period of time	t_p	[day]	= 30
Time increment	dt	[day]	= 0.1
Loading/Reloading consolidation ratio $\frac{C_v(NC)}{C_v(OC)}$	β	[-]	= 0.095

Impervious bottom boundary

The settlement is not required in this example, therefore any reasonable value for the coefficient of permeability may be defined.

The degree of consolidation under a rectangular cyclic loading at a period of time t_p as shown in Figure 5.70 is determined at different periods and tabulated against that of *Toufigh and Ouria* (2009).

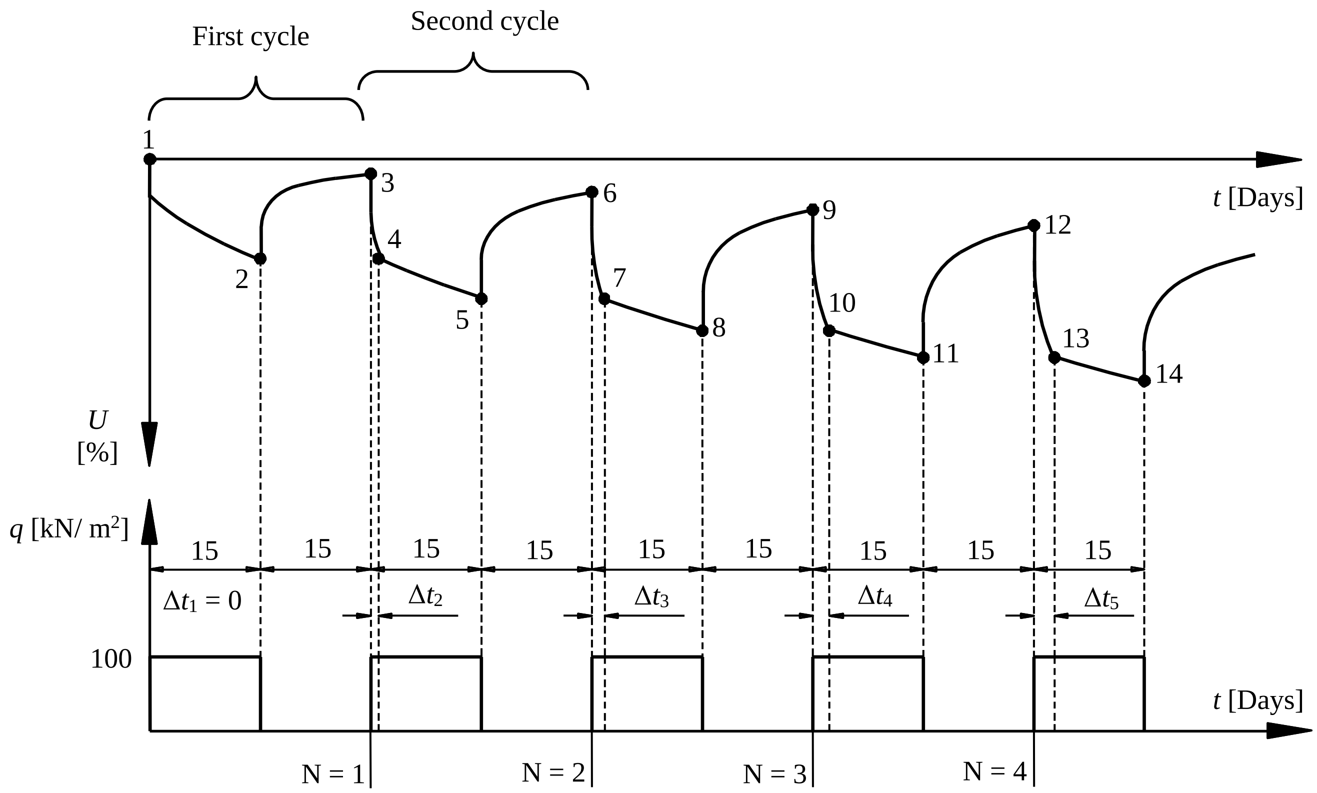


Figure 5.70 Rectangular cyclic loading scheme

5.7.1.4 Results:

Results of *LEM* in *GEO Tools* show a good agreement with those of *Toufigh and Ouria* (2009) for a rectangular cyclic loading at different periods of times as shown in Table 5.16.

Table 5.16 Comparison of the results obtained from *LEM* with those of *Toufigh and Ouria* (2009)

Half Cycle	Time *t* [days]	Degree of consolidation *U* [%]	
		Toufigh/ Ouria (2009)	*GEO Tools*
1	15	13.23	13.48
2	30	1.95	1.96
3	45	18.82	18.91
4	60	4.47	4.43
5	75	23.23	23.02
6	90	6.89	6.74
7	105	26.96	26.53
8	120	9.07	8.81
9	135	30.19	29.69
10	150	11.01	10.70
11	165	33.04	32.57

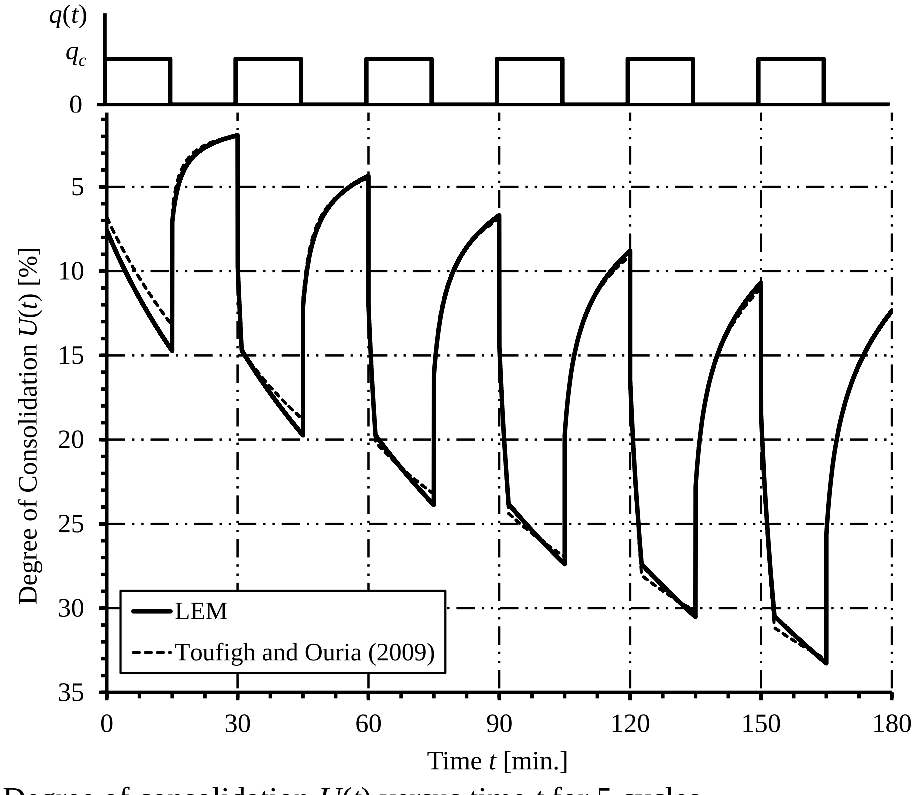


Figure 5.71 Degree of consolidation $U(t)$ versus time t for 5 cycles

5.7.1.5 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation of the consolidation under a rectangular cyclic loading are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of inelastic clays under rectangular cyclic loading
Date: 29-09-2017
Project:  Toufigh and Ouria 2009
File: Toufigh 2009

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Rectangular cyclic loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo         [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 1.80
Depth increment in z-direction       Di         [m]     = 0.18

Time:
Time of consolidation                Tr         [Days]  = 180.00
Time increment                       dT         [Days]  = 0.10
Time                                 T1         [Days]  = 0.00
Time                                 T2         [Days]  = 15.00
Time                                 T3         [Days]  = 0.00
Time                                 T4         [Days]  = 15.00
Period of time                       Tp         [Days]  = 30.00
No. of periods                       Np         [Days]  = 6

No. of time intervals                Nt         [-]     = 1800
Time interval                        Ti         [Days]  = 0.10

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1       1.80         10      3.3565E-08      1.0000E-09
------------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.095
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 1.000

Results:
Degree of consolidation                       Up   [%]  = 12.42
Degree of consolidation                       Us   [%]  = 12.42
Settlement                                    s    [cm] = 6.7922
```

Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	100.00	0.00
1	0.18	100.00	-3.49
2	0.36	100.00	-6.80
3	0.54	100.00	-9.77
4	0.72	100.00	-12.29
5	0.90	100.00	-14.31
6	1.08	100.00	-15.83
7	1.26	100.00	-16.90
8	1.44	100.00	-17.58
9	1.62	100.00	-17.96
10	1.80	100.00	-18.08

Loading type:

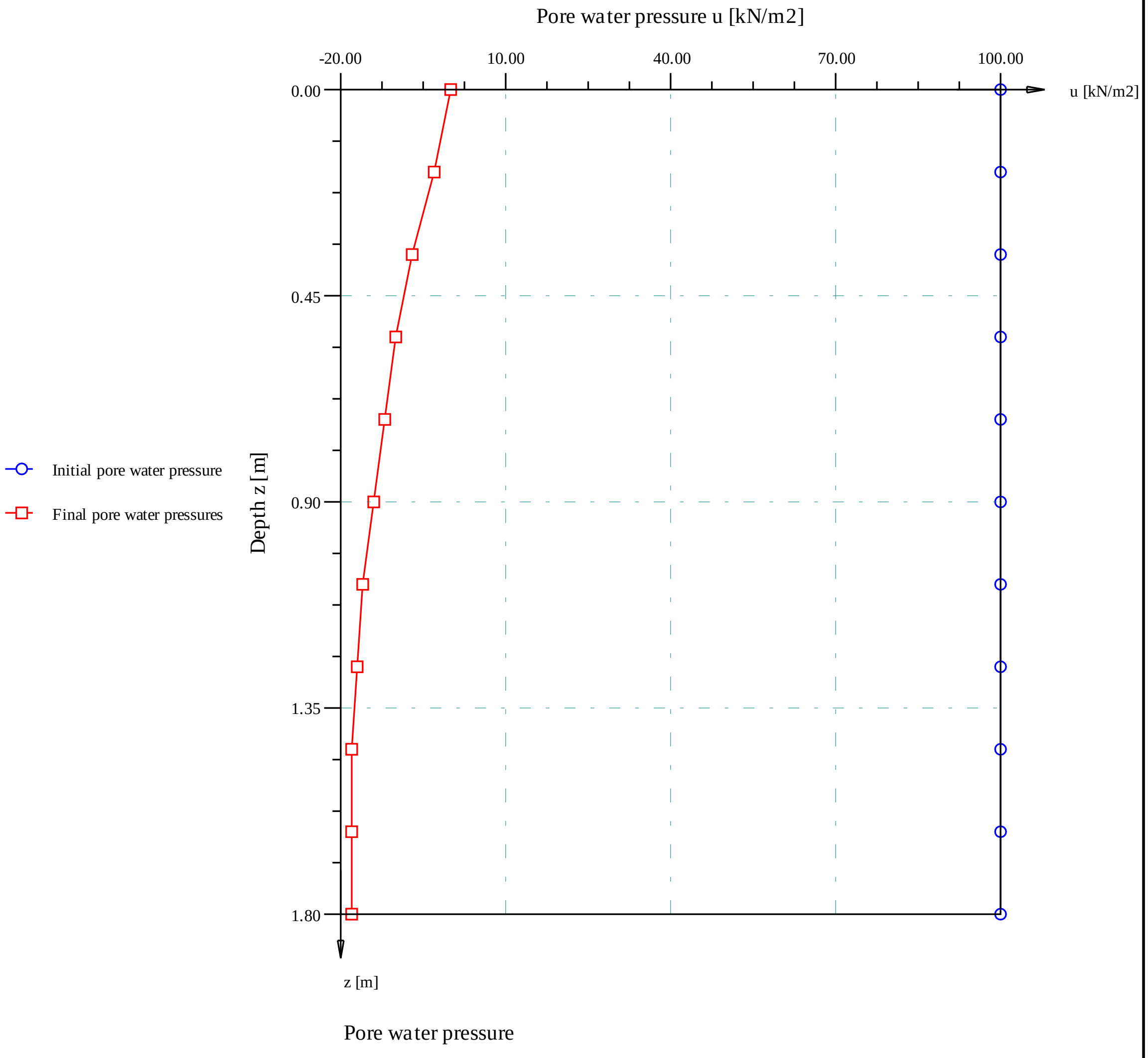
No. I [-]	Time T [Days]	Degree of consolidation U [%]	Loading type
1	15.00	13.48	Loading
2	31.10	13.91	Reloading
3	45.00	18.91	Loading
4	61.70	18.98	Reloading
5	75.00	23.02	Loading
6	92.20	23.07	Reloading
7	105.00	26.53	Loading
8	122.70	26.69	Reloading
9	135.00	29.69	Loading
10	153.20	29.95	Reloading

Pore water pressure U [kN/m2]:

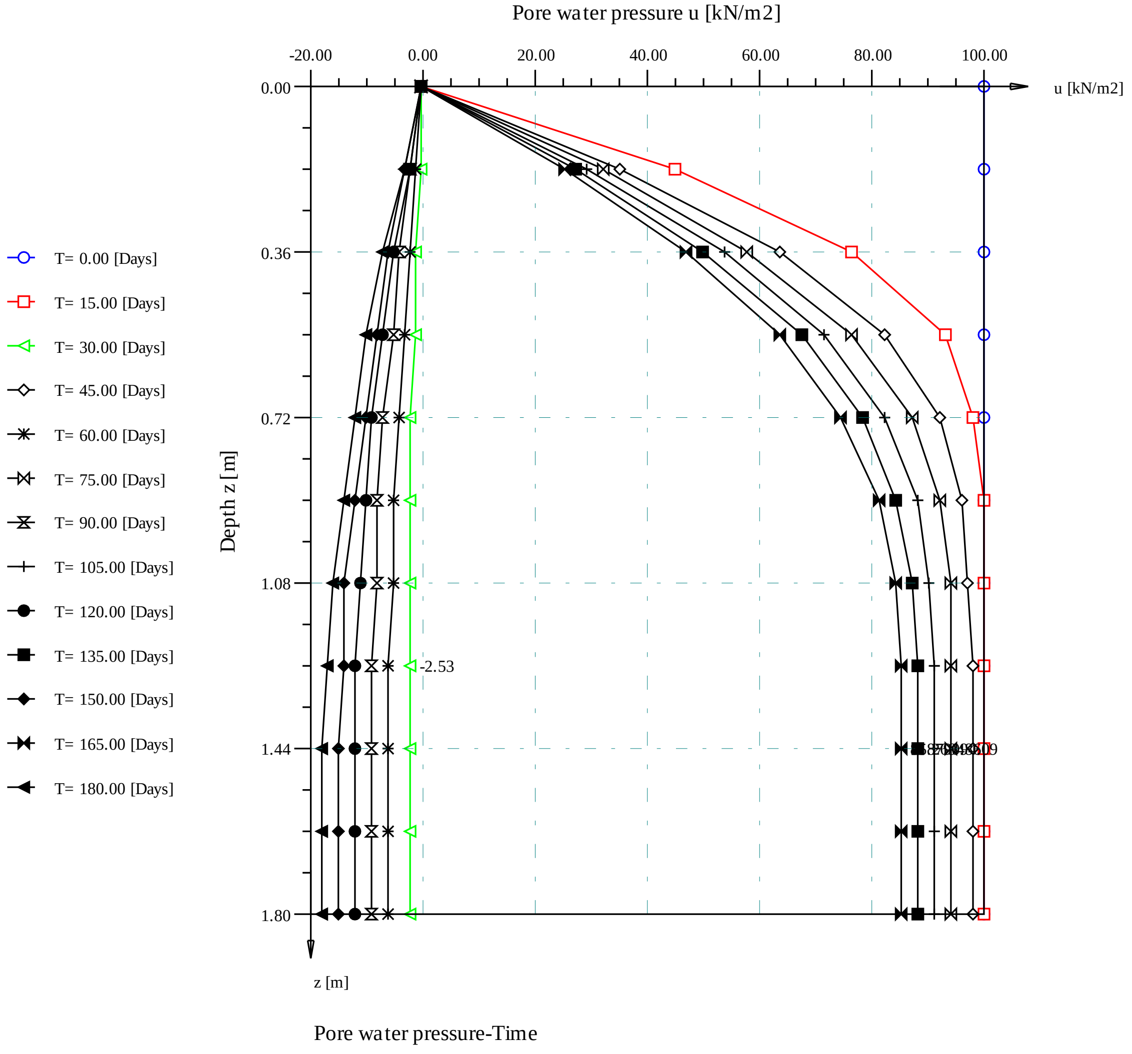
T [Days] \ z [m]	0.00	15.00	30.00	45.00	60.00	75.00	90.00	105.00	120.00	135.00	150.00	165.00	180.00
0.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.18	100.00	44.56	-0.67	35.43	-1.38	31.58	-1.99	28.92	-2.54	26.70	-3.03	24.81	-3.49
0.36	100.00	76.32	-1.28	63.94	-2.66	57.93	-3.87	53.57	-4.93	49.83	-5.91	46.58	-6.80
0.54	100.00	92.44	-1.79	82.37	-3.76	76.24	-5.51	71.43	-7.06	67.15	-8.47	63.32	-9.77
0.72	100.00	98.23	-2.16	91.92	-4.63	86.78	-6.86	82.37	-8.83	78.29	-10.63	74.51	-12.29
0.90	100.00	99.70	-2.39	95.85	-5.25	91.73	-7.88	87.93	-10.22	84.33	-12.35	80.93	-14.31
1.08	100.00	99.96	-2.51	97.12	-5.65	93.58	-8.60	90.19	-11.23	87.00	-13.63	83.98	-15.83
1.26	100.00	100.00	-2.53	97.43	-5.88	94.06	-9.06	90.83	-11.91	87.85	-14.51	85.07	-16.90
1.44	100.00	100.00	-2.50	97.50	-5.97	94.09	-9.32	90.86	-12.33	87.94	-15.07	85.26	-17.58
1.62	100.00	100.00	-2.47	97.51	-6.01	94.05	-9.44	90.76	-12.55	87.83	-15.37	85.18	-17.96
1.80	100.00	100.00	-2.46	97.52	-6.01	94.03	-9.48	90.71	-12.61	87.77	-15.46	85.12	-18.08

Degree of consolidation/ Settlement:

T [Days]	15.00	30.00	45.00	60.00	75.00	90.00	105.00	120.00	135.00	150.00	165.00
Us[%]	13.48	1.96	18.91	4.43	23.02	6.74	26.53	8.81	29.69	10.70	32.57



GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1000	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**



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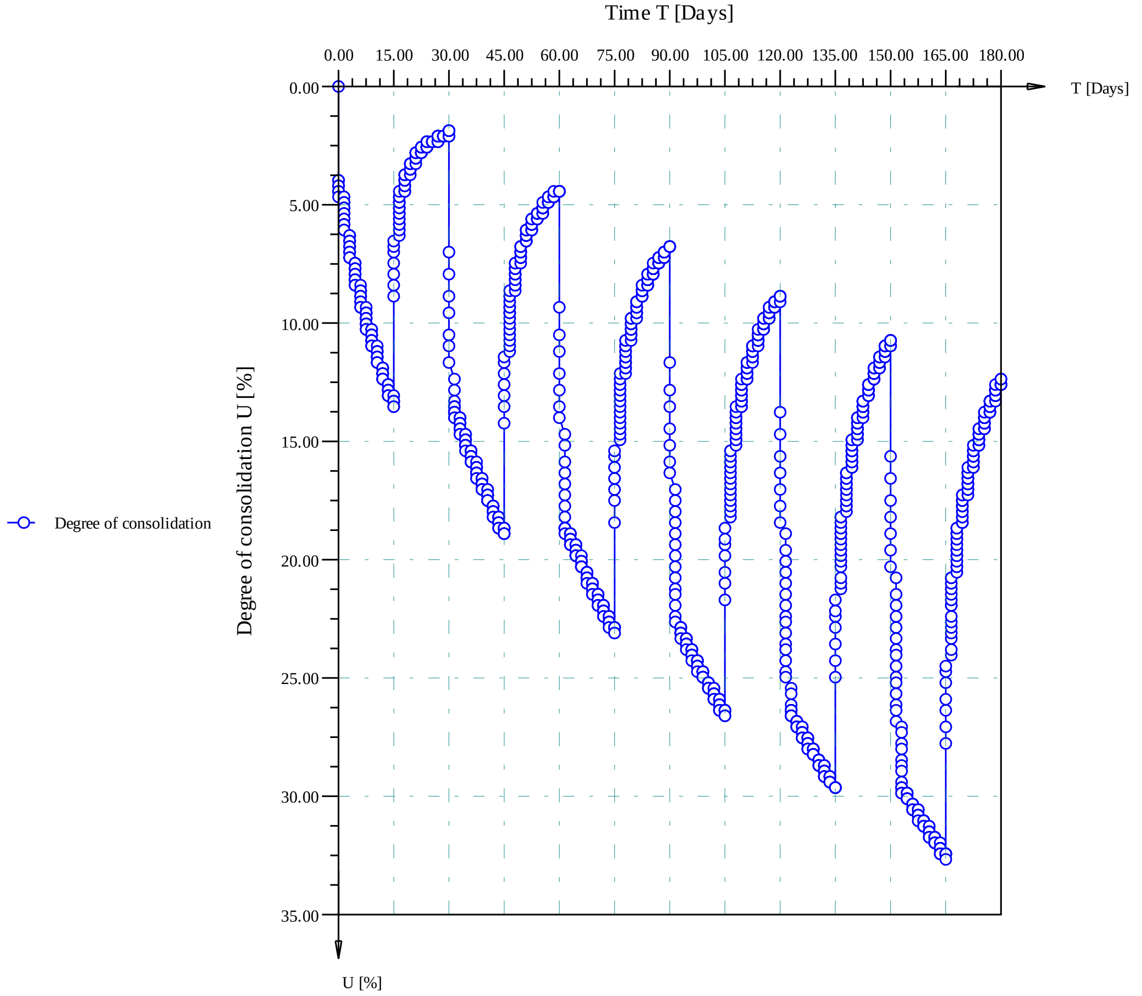
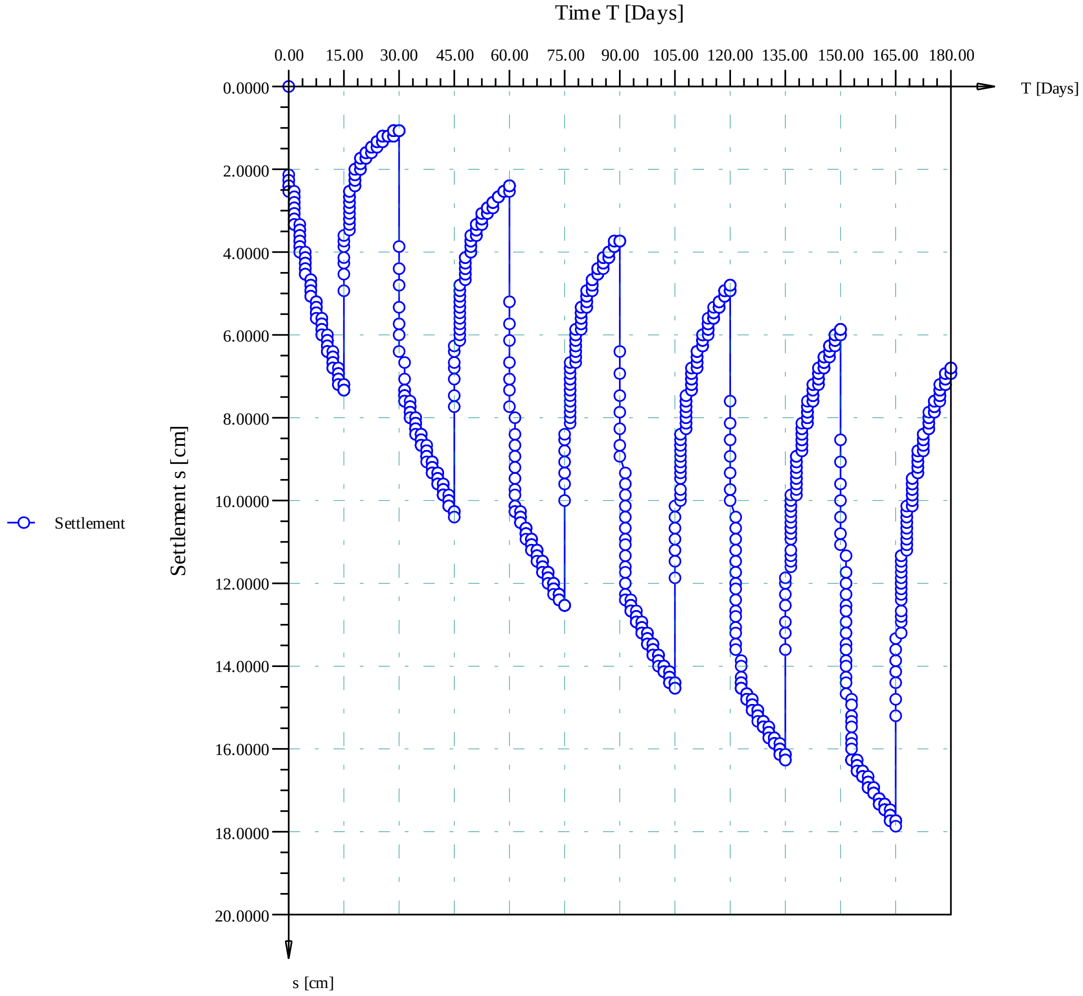
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 982	Project: Toufigh and Ouria 2009
File: Toufigh 2009	Date: 29-09-2017
Page No.:	Title: Consolidation of inelastic clays under rectangular cyclic loading



Degree of consolidation-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1500	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**



Settlement-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1500	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009	**Date: 29-09-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**

5.7.2 Example 13: Excess Pore Water Pressure due to Rectangular Cyclic Loading

5.7.2.1 Description of the problem

Another analytical solution is presented by *Conte* and *Troncone* (2006) to be compared with their procedure. The present case was considered by *Baligh* and *Levadoux* (1978), who proposed an analytical solution to calculate the excess pore water pressure at any depth and at the end of a given number, N_h, of half-cycles of loading.

A single drainage elastic clay layer is subjected to a rectangular cyclic load of intensity q_c and a period $T = 0.1H^2/ C_v$ as shown in Figure 5.72.

5.7.2.2 Data:

Initial pore water pressure	u_o	[kN/m²]	= 1.0
Total layer thickness	H_d	[m]	= 1.0
Depth increment in z-direction	D_i	[m]	= 0.1
Coefficient of consolidation	C_v	[m²/ sec]	= 1.157×10⁻⁵
Coefficient of permeability	k_v	[m/ sec]	= 1.0×10⁻⁵
Period of time	t_p	[days]	= 0.1
Time increment	dt	[days]	= 0.01
Loading/Reloading consolidation ratio $\frac{C_v(NC)}{C_v(OC)}$	β	[-]	= 0.999 (assumed ≈1.0)

Impervious bottom boundary

5.7.2.3 Analysis of the problem

To examine the accuracy of the numerical analysis of consolidation of clay under rectangular cyclic load using the *LEM* method, the clay layer is divided into 10 equal layers each of depth 0.1*H* and the time is divided into 8000 intervals each of 0.0025*T*.

The settlement is not required in this example, therefore any reasonable value for the coefficient of permeability may be defined.

5.7.2.4 Results and discussions

Pore water pressure ratios *u/q* with depth ratio *z/H* at different half-cycles *N* obtained from *LEM* were compared in Figure 5.73 with those obtained by *Baligh* and *Levadoux* (1978) and *Conte* and *Troncone* (2006). The figure shows a good agreement with the analytical results of *Baligh* and *Levadoux* (1978) and with the procedure of *Conte* and *Troncone* (2006).

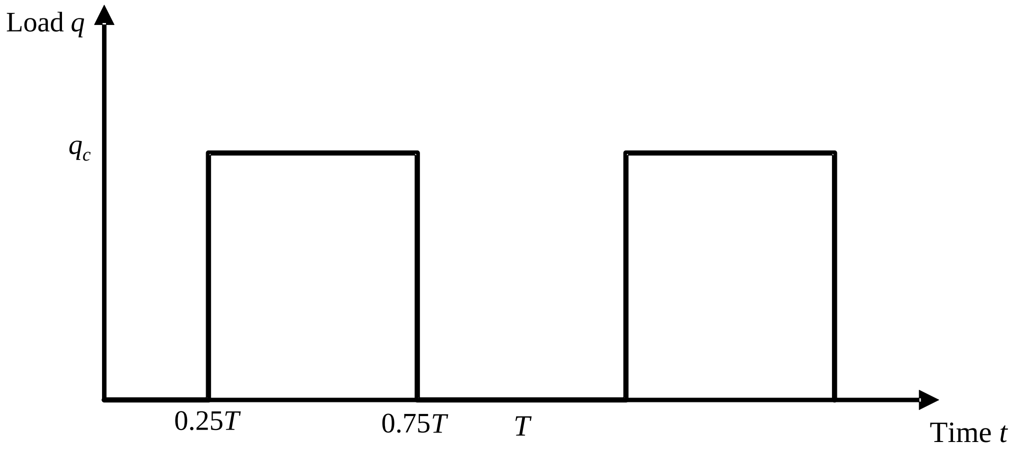


Figure 5.72 Loading scheme used in the present verification

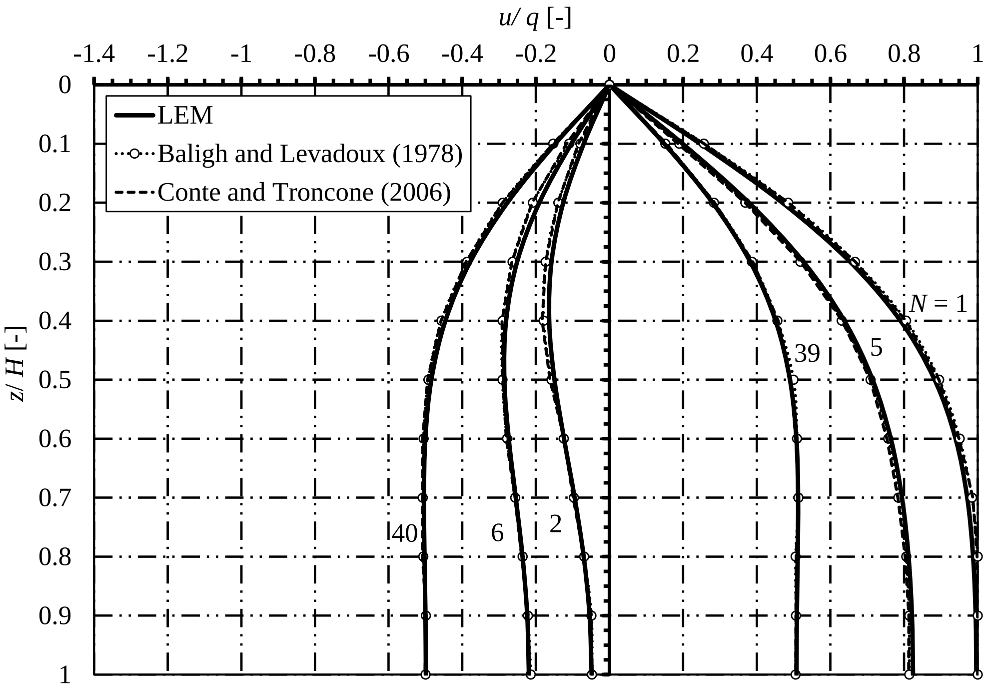


Figure 5.73 Pore water pressure ratios at different number of loading half-cycles, N_h

5.7.2.5 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation of the consolidation under a rectangular cyclic loading are presented on the next pages.

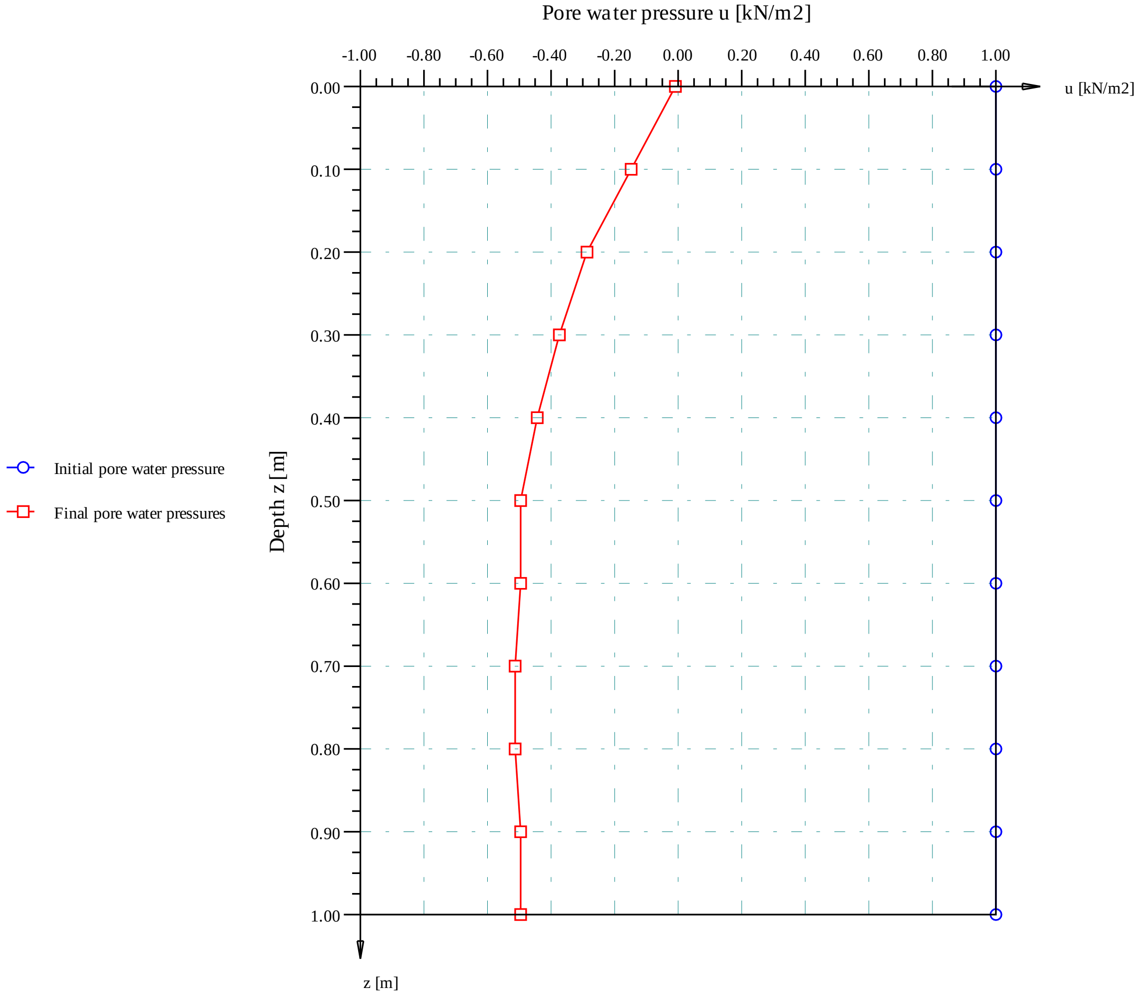
```
**********************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
**********************************************************
Title: Consolidation theory for cyclic loading, ASCE
Date: 06-10-2017
Project: Baligh and Levadoux (1978)
File: Baligh 1978

----------------------------------------------------------
Degree of consolidation
----------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Rectangular cyclic loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 1.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 1.00
Depth increment in z-direction       Di          [m]     = 0.10

Time:
Time of consolidation                Tr          [Days]  = 4.00
Time increment                       dT          [Days]  = 0.01
Time                                 T1          [Days]  = 0.00
Time                                 T2          [Days]  = 0.05
Time                                 T3          [Days]  = 0.00
Time                                 T4          [Days]  = 0.05
Period of time                       Tp          [Days]  = 0.10
No. of periods                       Np          [Days]  = 40

No. of time intervals                Nt          [-]     = 400
Time interval                        Ti          [Days]  = 0.01

Boring:
---------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
---------------------------------------------------------
    1       1.00         10      1.1574E-05      1.0000E-05
---------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.999
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 1.000
```

Results:

Degree of consolidation	Up	[%]	= 40.27
Degree of consolidation	Us	[%]	= 40.27
Settlement	s	[cm]	= 3.55

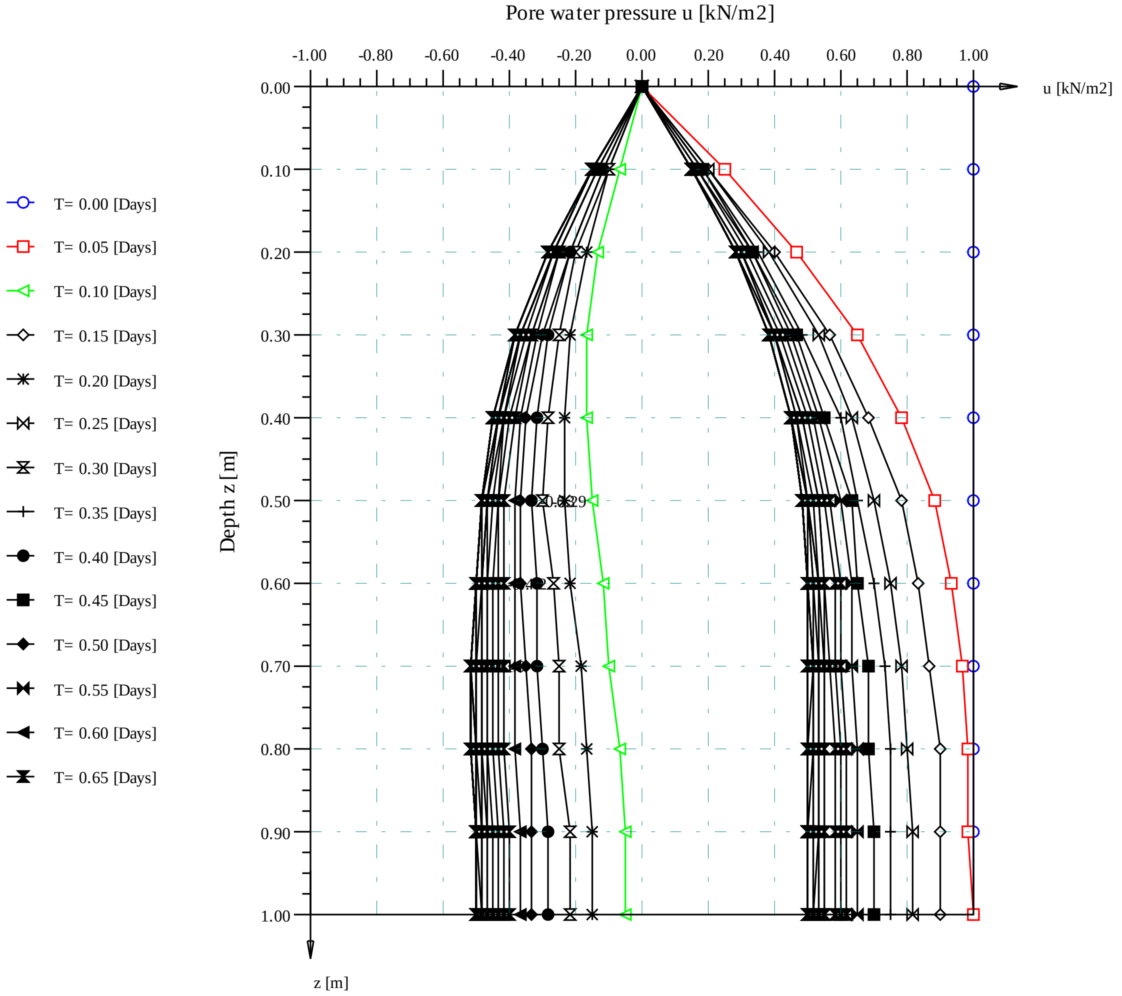
Initial and Final pore water pressures with depth:

No. I [-]	Depth z [m]	Initial pore water pressure uo [kN/m2]	Final pore water pressures uf [kN/m2]
0	0.00	1.00	0.00
1	0.10	1.00	-0.15
2	0.20	1.00	-0.28
3	0.30	1.00	-0.38
4	0.40	1.00	-0.45
5	0.50	1.00	-0.49
6	0.60	1.00	-0.50
7	0.70	1.00	-0.51
8	0.80	1.00	-0.51
9	0.90	1.00	-0.50
10	1.00	1.00	-0.50



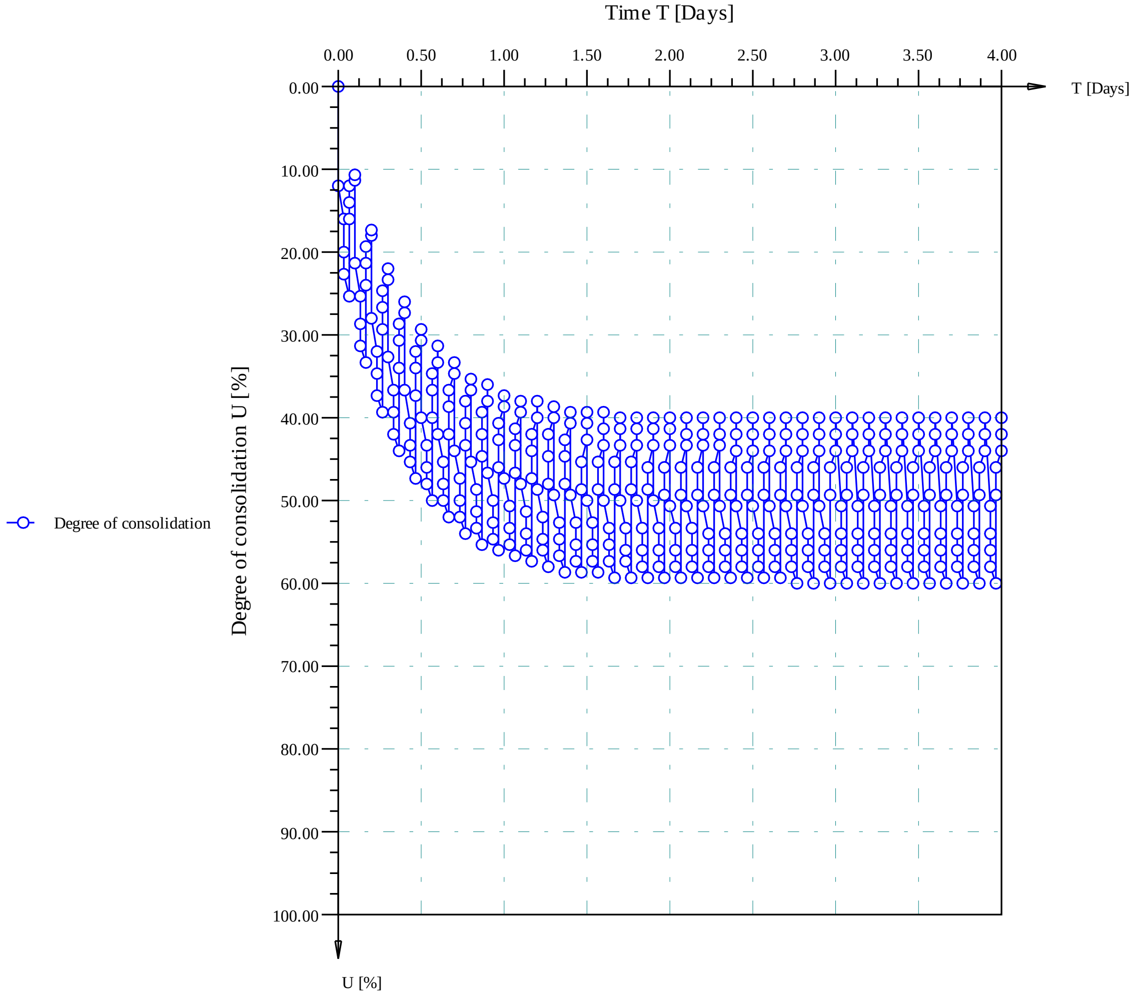
Pore water pressure

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 17	**Project: Baligh and Levadoux (1978)**
File: Baligh 1978	**Date: 06-10-2017**
Page No.:	**Title: Consolidation theory for cyclic loading, ASCE**



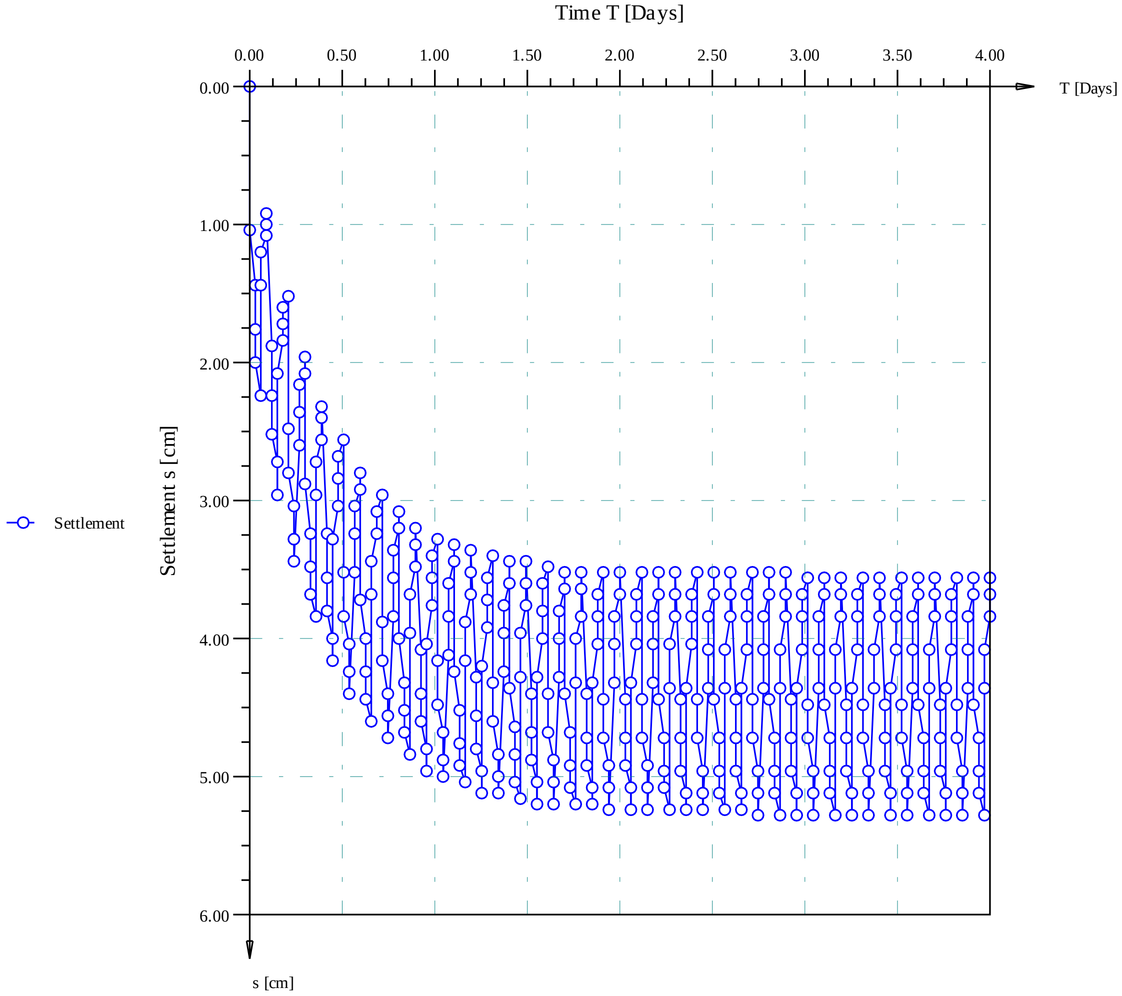
Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 17	**Project: Baligh and Levadoux (1978)**
File: Baligh 1978	**Date: 06-10-2017**
Page No.:	**Title: Consolidation theory for cyclic loading, ASCE**



Degree of consolidation-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 33	**Project: Baligh and Levadoux (1978)**
File: Baligh 1978	**Date: 06-10-2017**
Page No.:	**Title: Consolidation theory for cyclic loading, ASCE**



Settlement-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 30	**Project: Baligh and Levadoux (1978)**
File: Baligh 1978	**Date: 06-10-2017**
Page No.:	**Title: Consolidation theory for cyclic loading, ASCE**

5.7.3 Example 14: Consolidation under Trapezoidal Cyclic Loading

5.7.3.1 Description of the problem:

Zhuang and *Xie* (2005) presented a semi-analytical method to solve consolidation problem by taking into consideration the variation of compressibility of soil under cyclic loading. In the method, soil stratum is divided equally into n layers, while load is also divided into small parts and time into intervals. The problem of consolidation of soil stratum under cyclic loading can then be dealt with at each time interval as linear consolidation of multi-layered soils under constant loading. To verify *LEM* in *GEO Tools* for the consolidation under a trapezoidal cyclic loading, the degree of consolidation U under a trapezoidal cyclic loading calculated by *Zhuang and Xie* (2005) is compared with that obtained by *LEM*. The presented test problem consists of a single drainage clay layer subjected to trapezoidal cyclic loading.

There are three dimensionless variables, $T_{vo} = C_v t_o / H_b^2$, α_o and β_o, that govern the consolidation process in one cycle. T_{vo} reflects the influence of construction time t_o, α_o and β_o reflect the properties of loading as shown in Figure 5.74. Therefore, the consolidation behavior of the soil can be investigated by giving one of the variables different values while fixing the values of the other variables.

Figure 5.74 shows the load scheme used in the present verification, which is trapezoidal load begins with linear loading phase varies from zero to maximum load q_c along time $\alpha_o t_o$, then constant loading phase with time interval (t_o-2$\alpha_o t_o$), after that unloading phase along time interval equals the linear loading phase time ($\alpha_o t_o$) and finally no-loading phase extends through time interval of ($\beta_o t_o$- t_o). The cycles repeated every time period equals ($\beta_o t_o$).

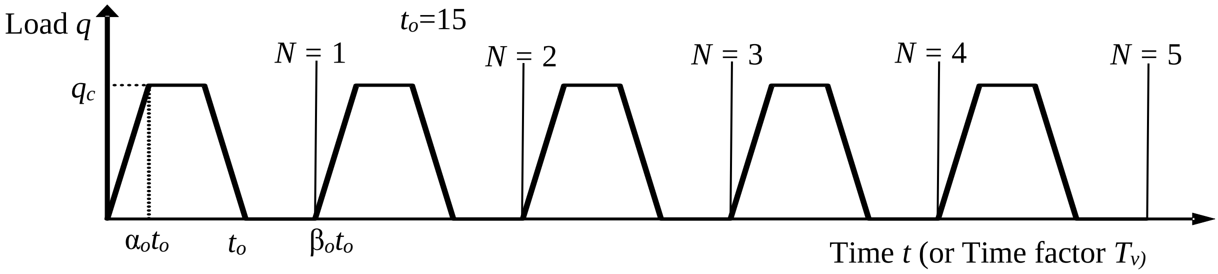


Figure 5.74 Loading scheme of trapezoidal cyclic loading

5.7.3.2 Data:

Initial pore water pressure	uo	[kN/m^2]	= 100
Total layer thickness	H_d	[m]	= 10
Depth increment in z-direction	Di	[m]	= 1
Coefficient of permeability	k_v	[m/ sec]	= 10^{-5}
Period of time	t_p	[days]	= 15
Loading/Reloading consolidation ratio $\dfrac{C_v(NC)}{C_v(OC)}$	β	[-]	= 0.999 (assumed ≈1.0)
No. of Periods	N_p	[-]	= 5

Time increment dt [days] =0.75

The time is divided into 100 intervals. The settlement is not required in this example, therefore any reasonable value for the coefficient of permeability may be defined.

5.7.3.3 Analysis of the problem

The degree of consolidation under a trapezoidal cyclic loading is determined at different periods and plotted against that of *Zhuang and Xie* (2005). Three cases of analyses are considered as shown in Table 5.17, in which the three dimensionless variables T_{vo}, α_c and β_o that govern the consolidation process in one cycle are chosen to be:

Time factor for one cycle $T_{vo} = C_v\, t_t/H_d^2$ =0.01, 0.1 and 1.0
Load geometry parameter α_o [-] = 0.1, 0.25, 0.4
Load geometry parameter β_o [-] = 1.1, 1.3, 1.5

Table 5.17 Cases of analyses

Variable			Case 1	Case 2	Case 3
Time of Construction	$t_{c=}\ \alpha_o t_o$	[Days]	3	1, 2.5, 4	3
Period of Time	$t_{b=}\ \beta_o\ t_o$	[Days]	15	15	11, 13, 15
Load geometry parameter	α_o	[-]	0.3	0.1, 0.25, 0.4	0.3
Load geometry parameter	β_o	[-]	1.5	1.5	1.1, 1.3, 1.5
Coefficient of consolidation	C_v	[m^2/ Day]	0.1, 1, 10	1	1

5.7.3.4 Results:

Results of *LEM* with those of *Zhuang and Xie* (2005) for a trapezoidal cyclic loading at different periods of times are presented in Figure 5.75 to Figure 5.76. From the figures, it's obvious that the *LEM* in *GEO Tools* gives good results for determining the degree of consolidation for clay subjected to trapezoidal cyclic loading.

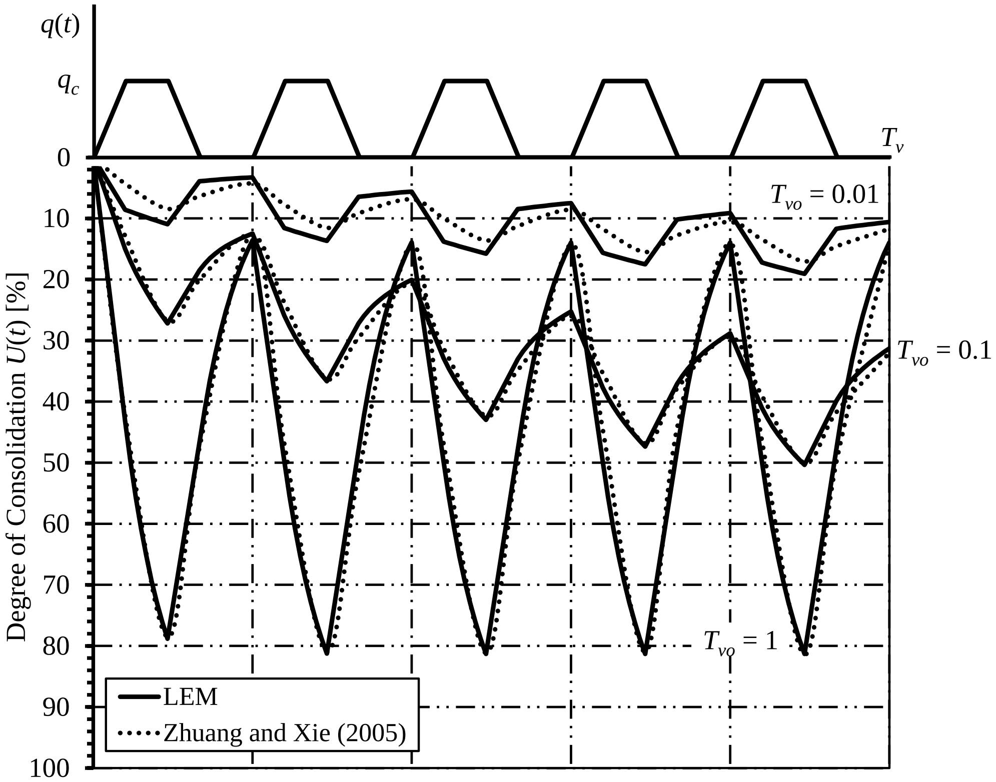


Figure 5.75 The influence of time factor T_v on degree of consolidation U

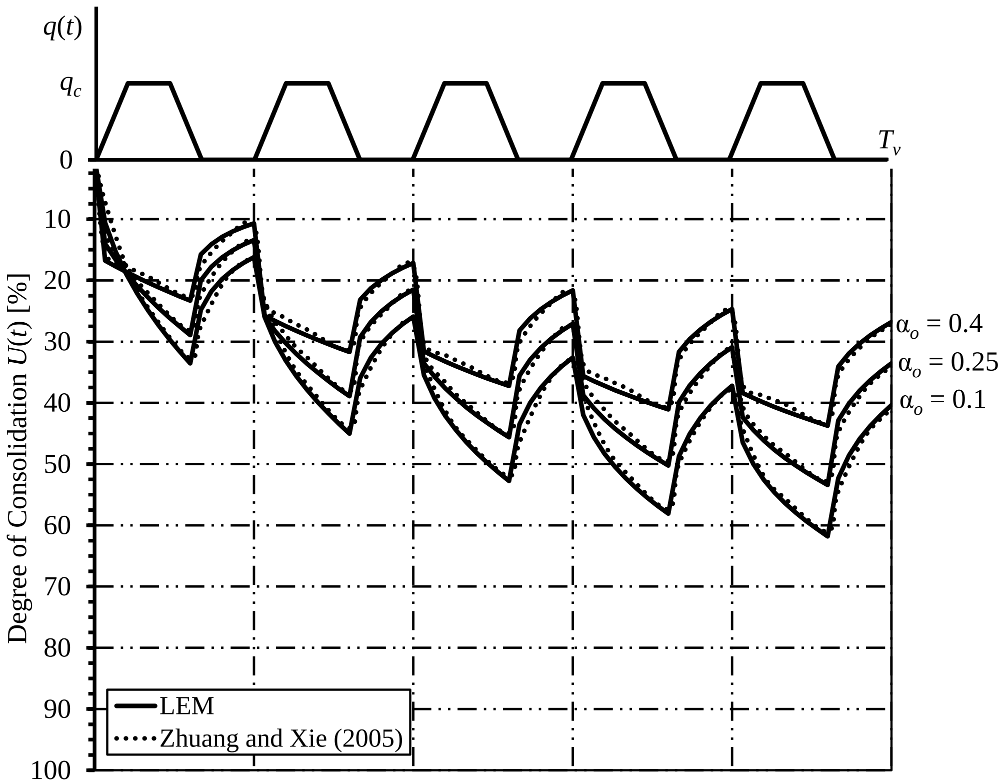


Figure 5.76 The influence of load parameter α_o on degree of consolidation U

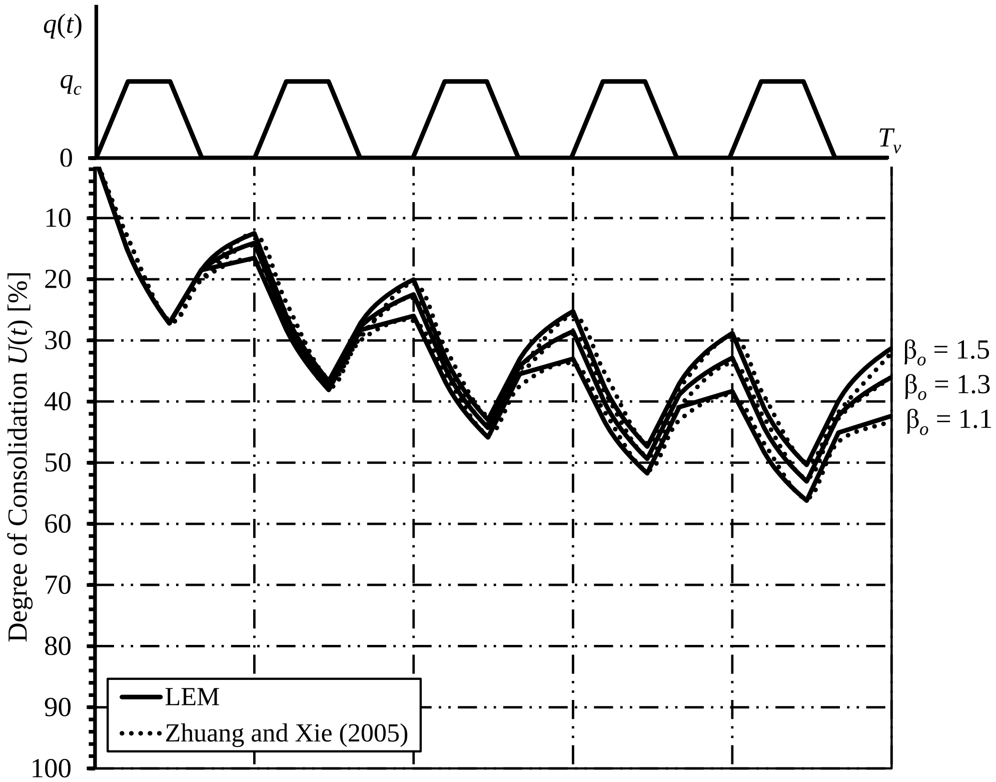


Figure 5.77 The influence of load parameter β_o on degree of consolidation U

5.7.3.5 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation of the consolidation under a trapezoidal cyclic loading for case 1 at $T_{vo} = 1$ are presented on the next pages.

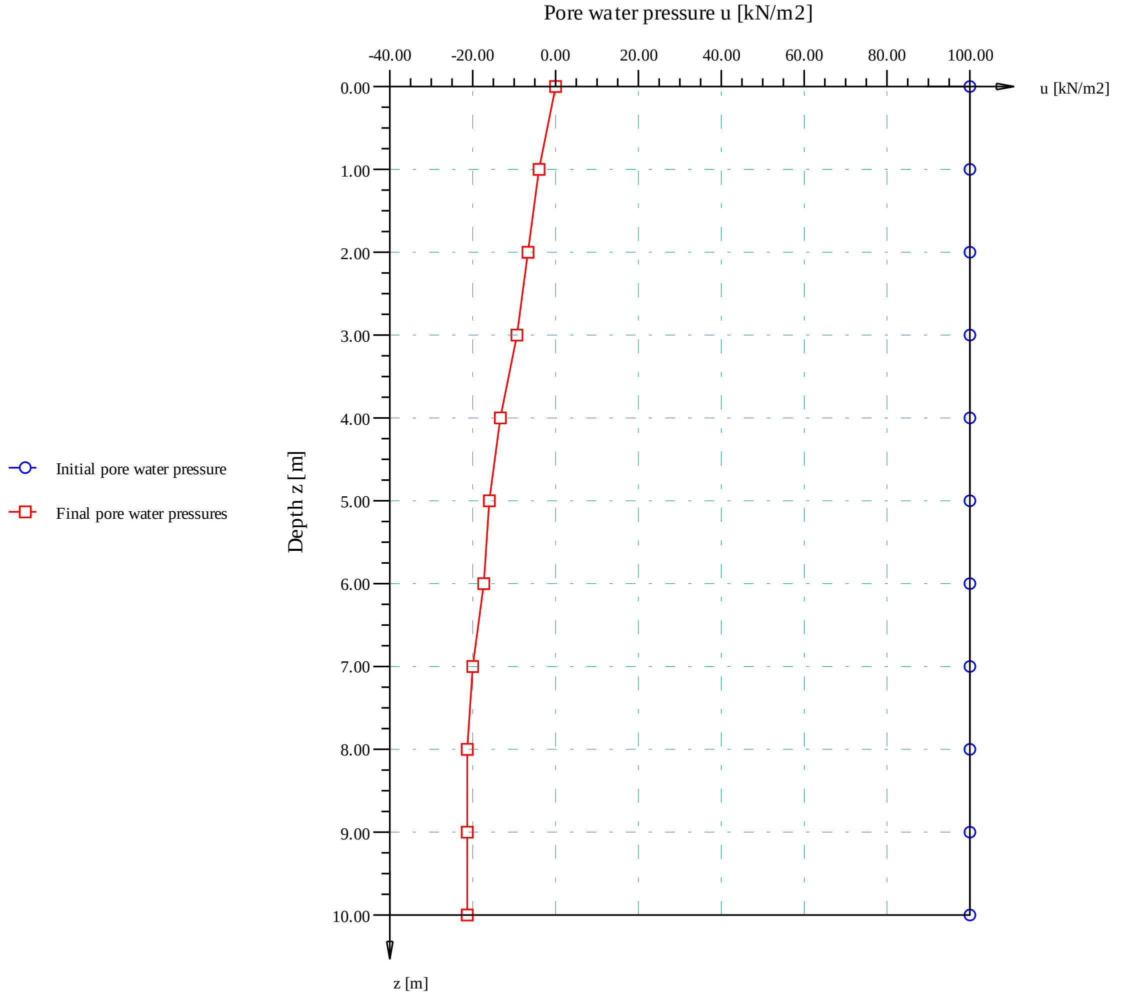
```
**********************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
**********************************************************
Title: Trapezoidal Cyclic Loading - Case 1 (Tvo=1)
Date: 15-09-2017
Project: Study on one-dimensional consolidation.... Zhuang and Xie (2005)
File: Zhuang (2005) Case 1

----------------------------------------------------------
Degree of consolidation
----------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Trapezoidal cyclic loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 100.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 10.00
Depth increment in z-direction       Di          [m]     = 1.00

Time:
Time of consolidation                Tr          [Days]  = 75.00
Time increment                       dT          [Days]  = 0.75
Time                                 T1          [Days]  = 3.00
Time                                 T2          [Days]  = 4.00
Time                                 T3          [Days]  = 3.00
Time                                 T4          [Days]  = 5.00
Period of time                       Tp          [Days]  = 15.00
No. of periods                       Np          [Days]  = 5


Boring:
---------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
---------------------------------------------------------
    1      10.00         10      1.1570E-04      1.0000E-07
---------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.999
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 1.000

Results:
Degree of consolidation                       Up   [%]  = 14.02
Degree of consolidation                       Us   [%]  = 14.02
Settlement                                    s    [cm] = 1.24
```

Initial and Final pore water pressures with depth:

No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
0	0.00	100.00	0.00
1	1.00	100.00	-3.44
2	2.00	100.00	-6.80
3	3.00	100.00	-10.00
4	4.00	100.00	-12.94
5	5.00	100.00	-15.57
6	6.00	100.00	-17.82
7	7.00	100.00	-19.62
8	8.00	100.00	-20.94
9	9.00	100.00	-21.75
10	10.00	100.00	-22.02

Loading type:

No.	Time	Degree of consolidation	Loading type
I	T	U	
[-]	[Days]	[%]	
1	7.00	78.69	Loading
2	22.00	81.15	Reloading
3	22.00	81.14	Loading
4	37.00	81.21	Reloading
5	37.00	81.20	Loading
6	52.00	81.21	Reloading
7	52.00	81.20	Loading
8	67.00	81.21	Reloading



Pore water pressure

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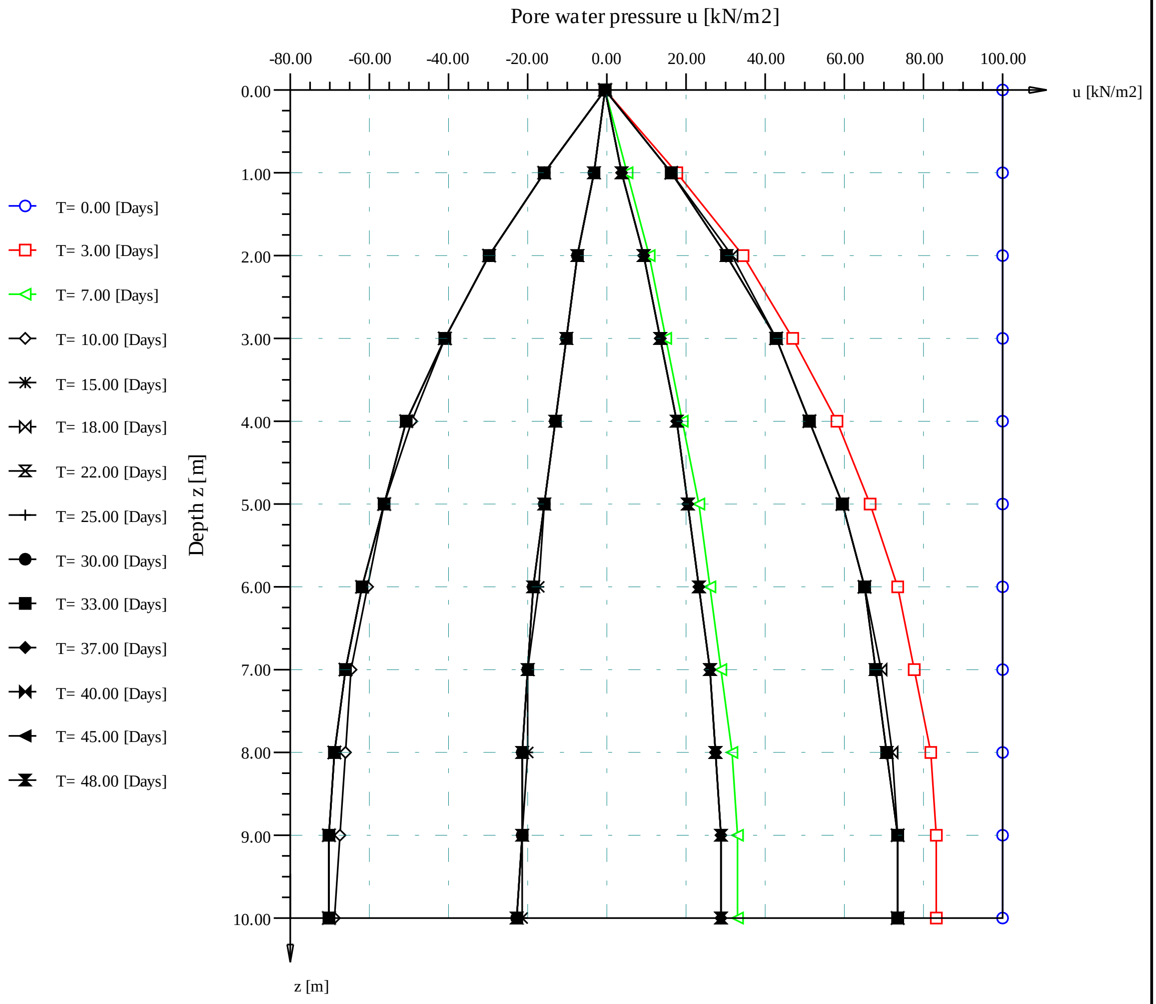
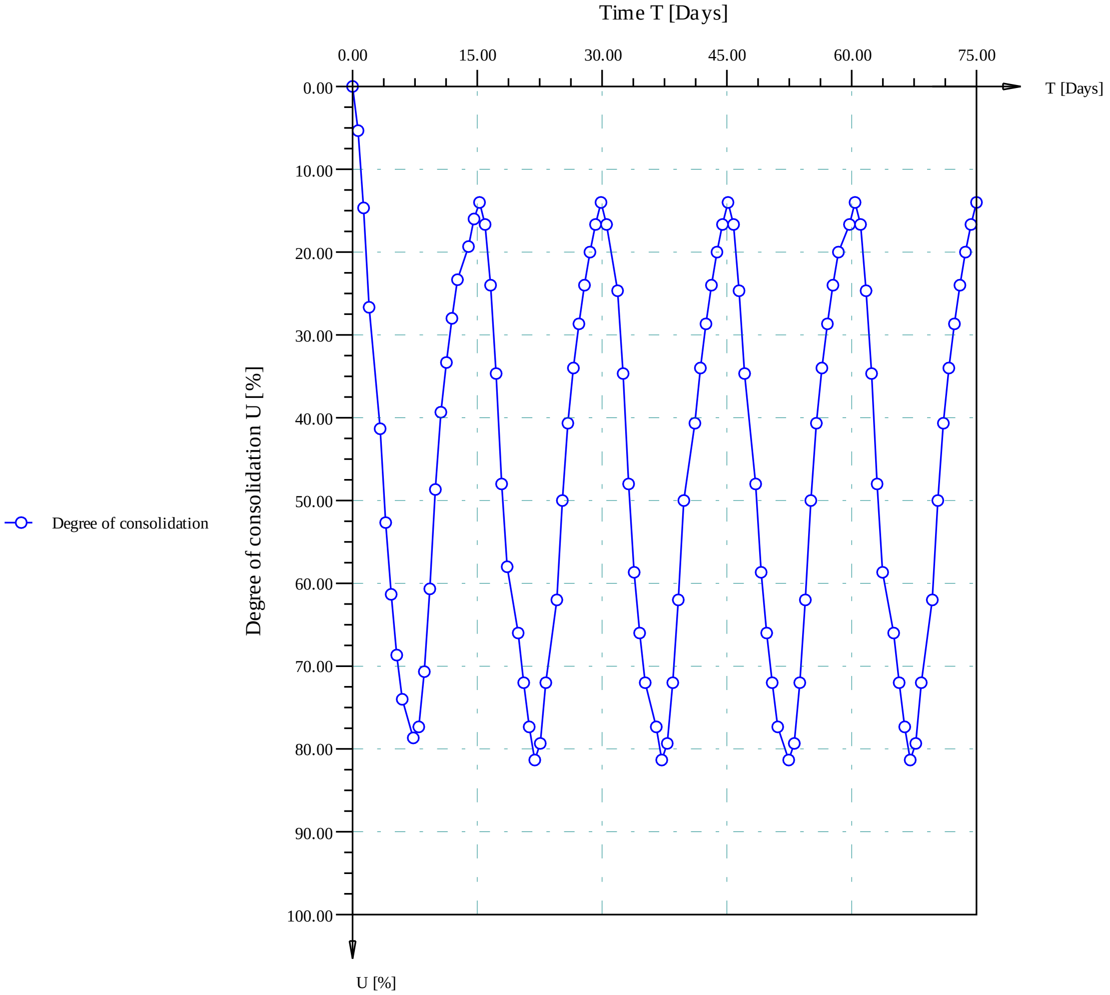
Scale: 1330	**Project: Study on one-dimensional consolidation.... Zhuang and Xie (2005)**
File: Zhuang (2005) Case 1	**Date: 15-09-2017**
Page No.:	**Title: Trapezoidal Cyclic Loading - Case 1 (Tvo=1)**



Pore water pressure-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 1400	**Project: Study on one-dimensional consolidation.... Zhuang and Xie (2005)**
File: Zhuang (2005) Case 1	**Date: 15-09-2017**
Page No.:	**Title: Trapezoidal Cyclic Loading - Case 1 (Tvo=1)**



Degree of consolidation-Time

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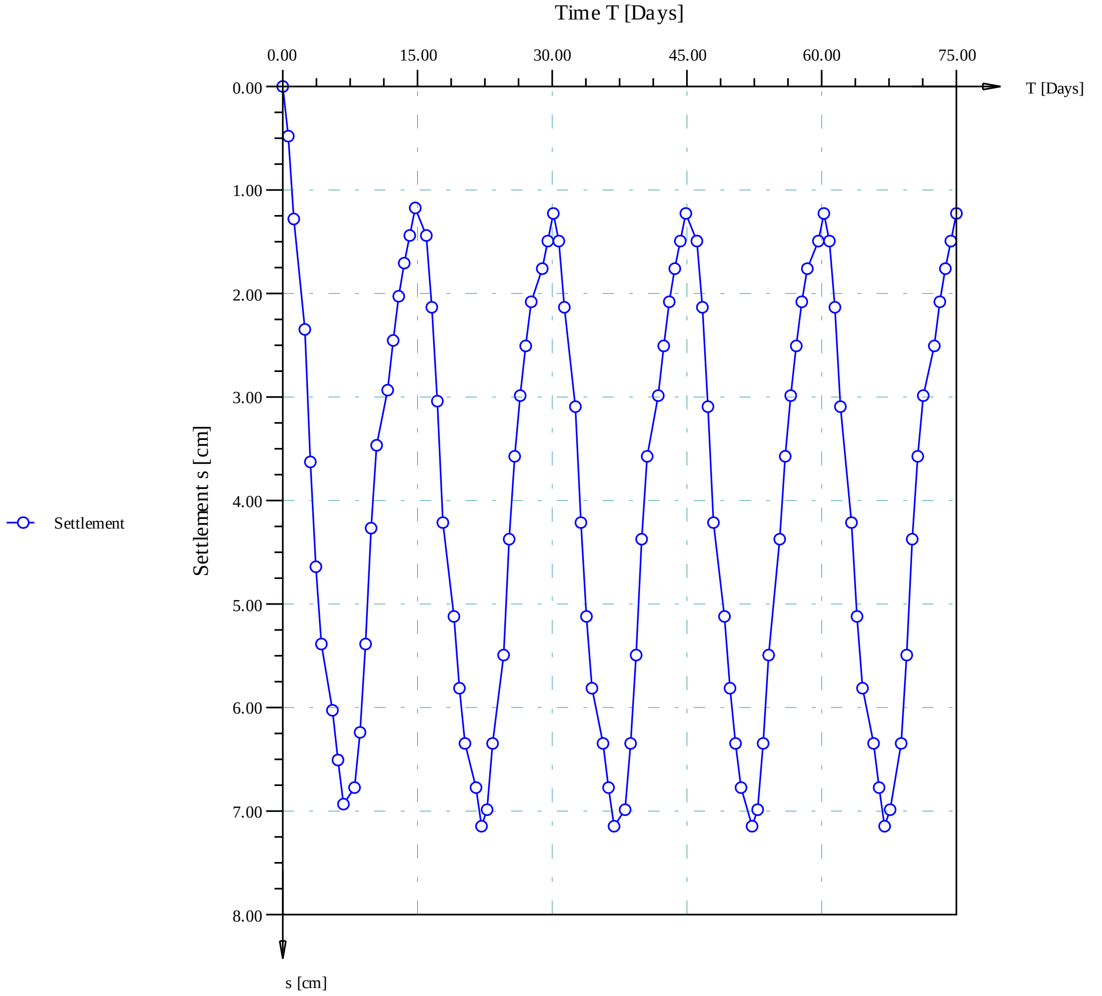
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 662	**Project: Study on one-dimensional consolidation.... Zhuang and Xie (2005)**
File: Zhuang (2005) Case 1	**Date: 15-09-2017**
Page No.:	**Title: Trapezoidal Cyclic Loading - Case 1 (Tvo=1)**



Settlement-Time

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Scale: 614	**Project: Study on one-dimensional consolidation.... Zhuang and Xie (2005)**
File: Zhuang (2005) Case 1	**Date: 15-09-2017**
Page No.:	**Title: Trapezoidal Cyclic Loading - Case 1 (Tvo=1)**

5.7.4 Example 15: Consolidation under Triangular Cyclic Loading

5.7.4.1 Description of the problem

Liu and *Griffiths* (2015) presented a general analytical solution for obtaining the excess pore water pressure in a consolidating layer due to depth and time-dependent changes of total stress. The solution of *Liu* and *Griffiths* (2015) was verified with three special cases, one of them were chosen to verify the *LEM* in *GEO Tools*. The chosen verification example was originally considered by *Liu et al.* (2015). *Liu et al.* (2015) considered a triangular cyclic load with a linearly varying total stress distribution with depth.

Liu et al. (2015) assumed single-drained condition of a clay layer has a total thickness of H = 10 [m] and coefficient of consolidation $C_v = 3 \times 10^{-6}$ [m^2/ sec.]. The variation of the stress with depth was assumed to be linear with maximum values at the top and bottom of the layer given by σ_t = 150 [kN/ m^2] and σ_b = 50 [kN/ m^2] respectively as shown in Figure 5.59Figure 5.78. Assume the coefficient of consolidation k_v= 1×10^{-08} [m^2/ sec].

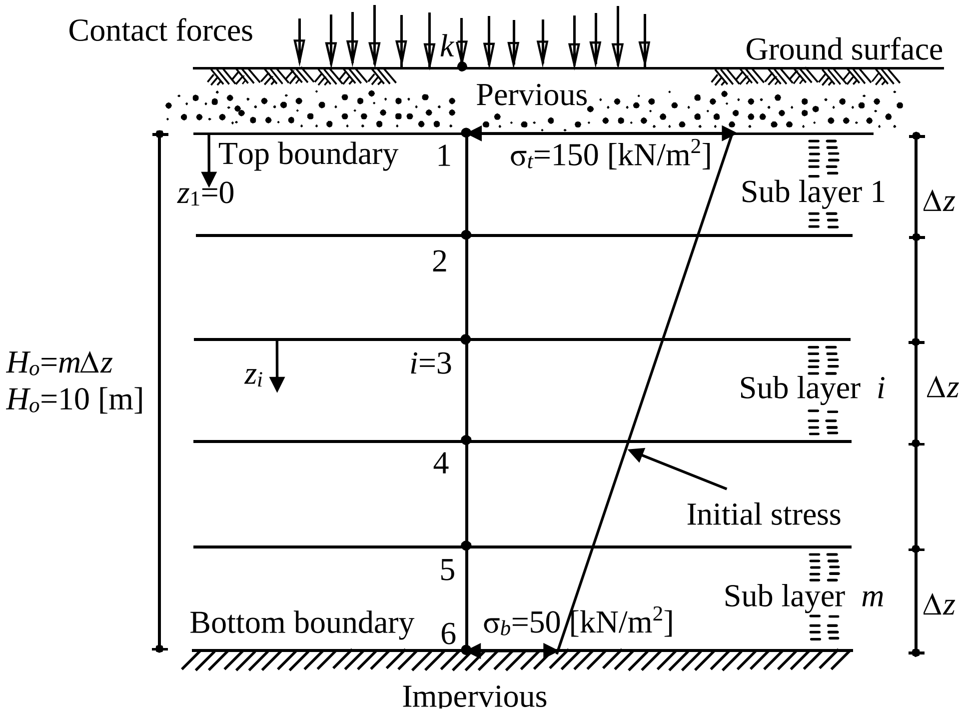


Figure 5.78 Initial excess pore water pressure on the clay layer

Figure 5.79 shows the load scheme used in the present verification, which is triangle load begins with linear loading phase varies from zero to maximum load q_c along time 0.25 years, then unloading phase along time interval equals the linear loading phase time 0.25 years. The cycles repeated every time period equals 0.5 years. The total time of cycling loadings is chosen to be 4 years.

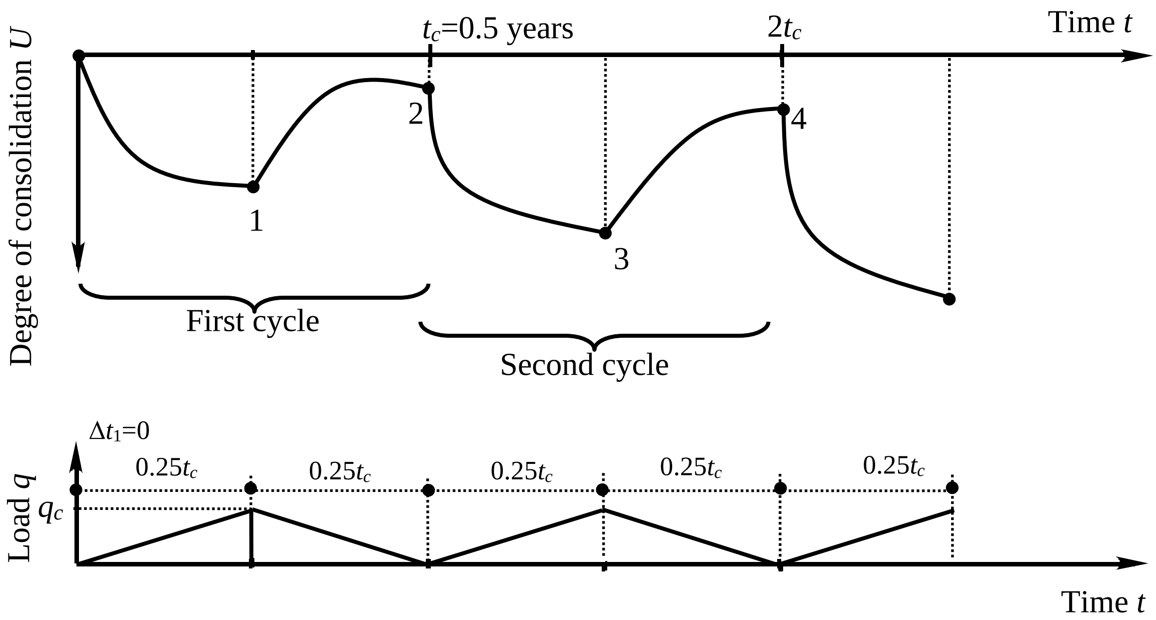


Figure 5.79 Loading scheme for triangular cyclic loading

5.7.4.2 Analysis of the problem

The clay layer is divided into 5 layers each of D_i=2 [m] thick, and the time is divided into 40 intervals, each of dt =0.1 year. Assume loading/reloading consolidation ratio $\frac{C_v(NC)}{C_v(OC)}$, β = 0.999 [-] ≈1.0

The settlement is not required in this example, therefore any reasonable value for the coefficient of permeability may be defined.

5.7.4.3 Results and discussions

As shown in Figure 5.80, which show the variation of excess pore water pressure with time at the layer base, the results of the *LEM* are identical with those of *Liu* and *Griffiths* (2015).

5.7.4.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* are presented on the next pages.

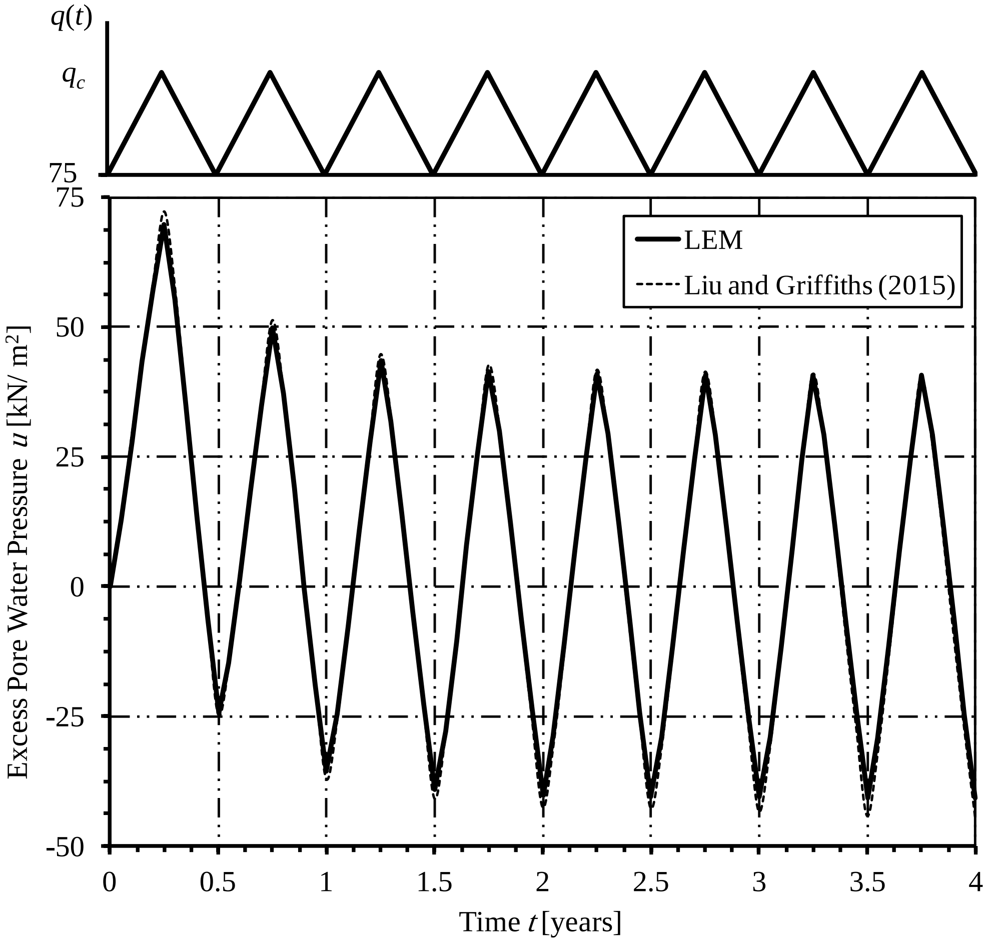


Figure 5.80 Variation of excess pore water pressure u with time t at the base of the layer

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: A general solution for 1D consolidation induced by depth- and time-
dependent changes in stress
Date: 29-09-2017
Project: Liu and Griffiths (2015)
File: Liu and Griffiths (2015)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Triangular cyclic loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Pore water pressure is defined by the user
Overburden pressure                  Po=Gamma*z [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb         [m]     = 10.00
Depth increment in z-direction       Di         [m]     = 2.00

Time:
Time of consolidation                Tr         [Years] = 4.00
Time increment                       dT         [Years] = 0.10
Time                                 T1         [Years] = 0.25
Time                                 T2         [Years] = 0.00
Time                                 T3         [Years] = 0.25
Time                                 T4         [Years] = 0.00
Period of time                       Tp         [Years] = 0.50
No. of periods                       Np         [Years] = 8

No. of time intervals                Nt         [-]     = 32
Time interval                        Ti         [Years] = 0.13

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1      10.00          5      3.0000E-06      1.0000E-08
------------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.999
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 1.000
```

Results:

Degree of consolidation	Up	[%]	= 36.36
Degree of consolidation	Us	[%]	= 36.36
Settlement	s	[cm]	= 12.35

Initial and Final pore water pressures with depth:

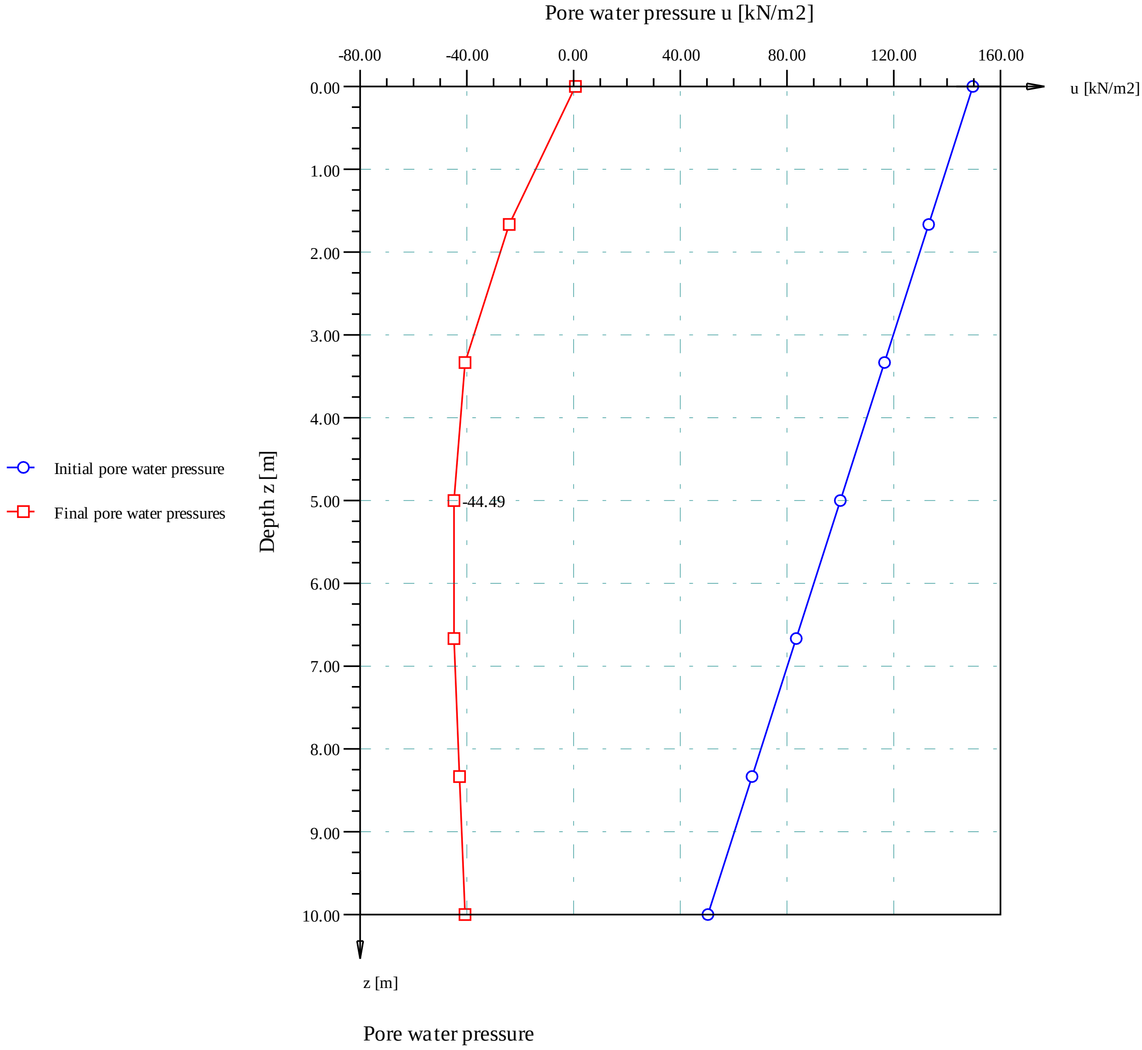
No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
0	0.00	150.00	0.00
1	1.67	133.33	-25.04
2	3.33	116.67	-39.61
3	5.00	100.00	-44.49
4	6.67	83.33	-44.34
5	8.33	66.67	-42.02
6	10.00	50.00	-40.66

Initial and Final pore water pressures with depth:

No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
1	0.00	150.00	0.00
2	10.00	50.00	-40.66

Loading type:

No.	Time	Degree of consolidation	Loading type
I	T	U	
[-]	[Years]	[%]	
1	0.25	45.02	Loading
2	0.75	57.86	Reloading
3	0.75	57.86	Loading
4	1.25	61.85	Reloading
5	1.25	61.84	Loading
6	1.75	63.09	Reloading
7	1.75	63.08	Loading
8	2.25	63.47	Reloading
9	2.25	63.46	Loading
10	2.75	63.59	Reloading
11	2.75	63.58	Loading
12	3.25	63.63	Reloading
13	3.25	63.62	Loading
14	3.75	63.64	Reloading



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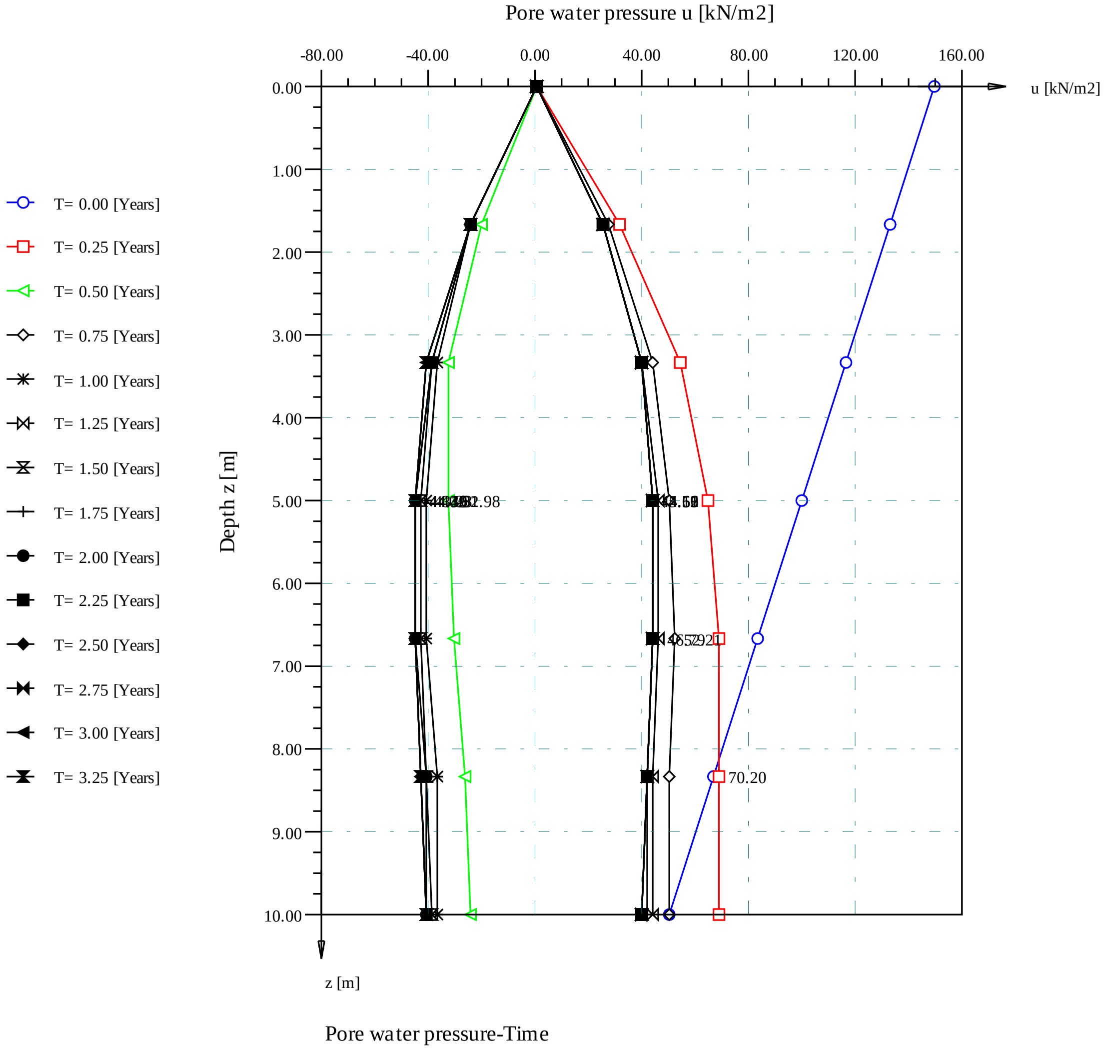
Scale: 2073	**Project: Liu and Griffiths (2015)**
File: Liu and Griffiths (2015)	**Date: 29-09-2017**
Page No.:	**Title: A general solution for 1D consolidation induced by depth- and time-dependent**



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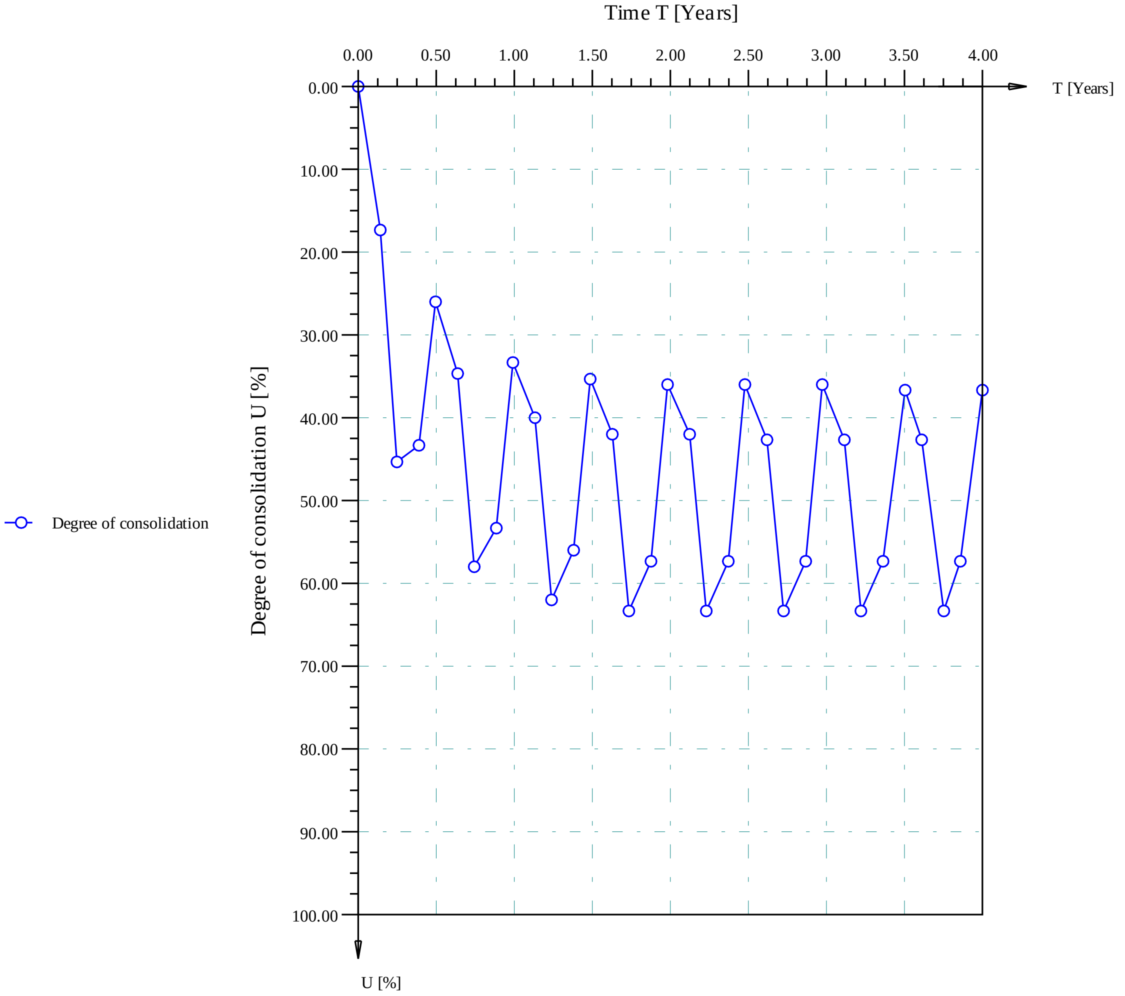
Scale: 2073	Project: Liu and Griffiths (2015)
File: Liu and Griffiths (2015)	Date: 29-09-2017
Page No.:	Title: A general solution for 1D consolidation induced by depth- and time-dependent



Degree of consolidation-Time

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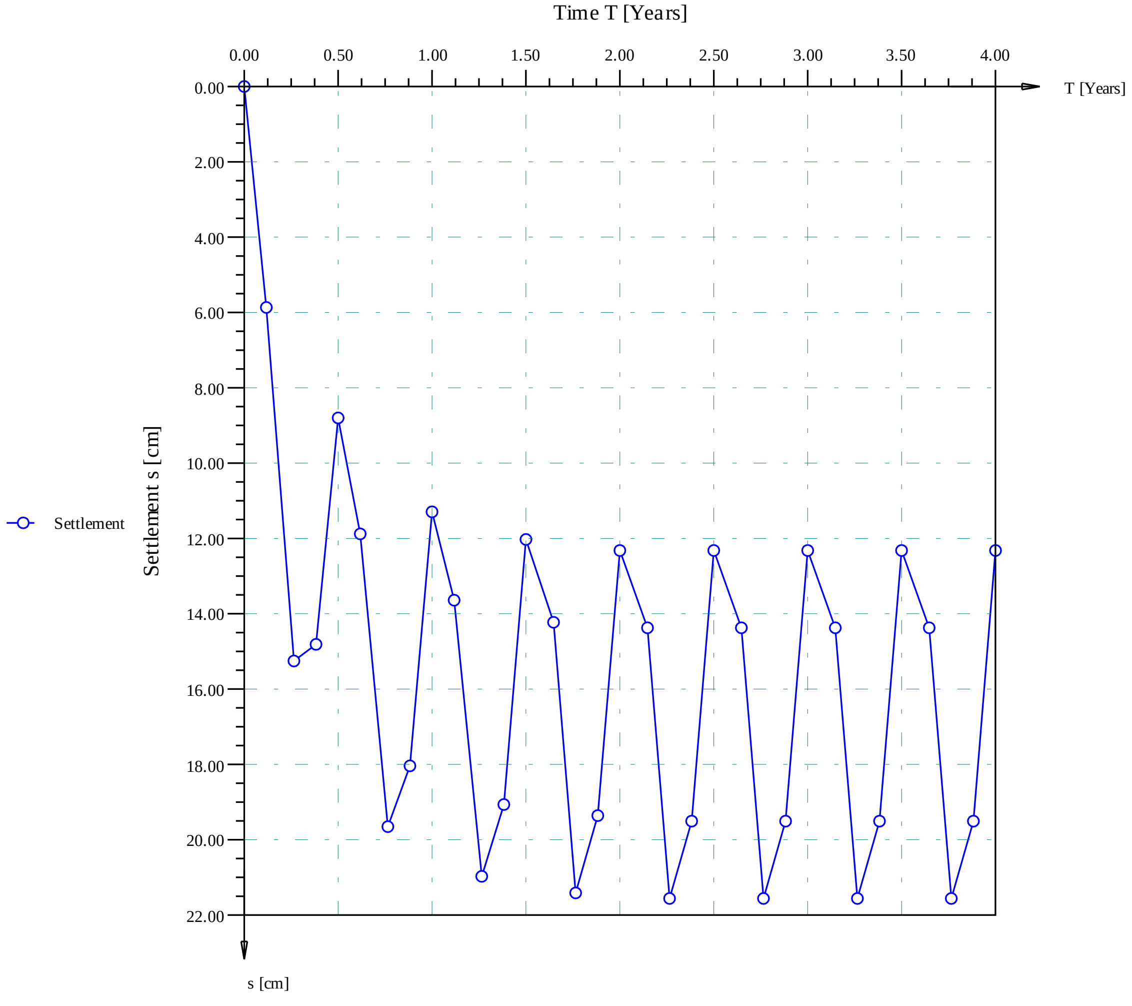
Scale: 36	**Project: Liu and Griffiths (2015)**
File: Liu and Griffiths (2015)	**Date: 29-09-2017**
Page No.:	**Title: A general solution for 1D consolidation induced by depth- and time-dependent**



Settlement-Time

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PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 29	**Project: Liu and Griffiths (2015)**
File: Liu and Griffiths (2015)	**Date: 29-09-2017**
Page No.:	**Title: A general solution for 1D consolidation induced by depth- and time-dependent**

5.7.5 Example 16: Settlement of Clay Subjected to Groundwater Level Oscillation

5.7.5.1 Description of the problem:

Ouria and *Toufigh* (2010) developed a finite element model based on *Biot's* three-dimensional consolidation theory and calibrated it by laboratory test results to predict the effect of groundwater table level oscillation on land subsidence in Kerman province in Iran. *Ouria* and *Toufigh* (2010) studied the groundwater table level oscillation in two zones, Nouq plain and in Rafsanjan plain. The results of Rafsanjan plain are chosen to verify the *LEM* in *GEO Tools*. The soil profile consists of a 150 [m] clay layer overlying by a 80 [m] layer of sand. In this case, 1 [m] groundwater table level oscillation with 1 [Year] period caused 8 [kN/ m^2] rectangular cyclic stress on the surface of the clay layer.

5.7.5.2 Data

Initial pore water pressure	u_o	[kN/m^2]	= 8
Total layer thickness	H_d	[m]	= 150
Depth increment in z-direction	D_i	[m]	= 5
Coefficient of consolidation	C_v	[m^2/ sec]	= 2.0602×10^{-5}
Coefficient of permeability	k_v	[m/ sec]	=2.0083×10^{-8}
Period of time	t_p	[years]	=1
Time increment	dt	[years]	=0.01
Loading/Reloading consolidation ratio $\dfrac{C_v(NC)}{C_v(OC)}$	β	[-]	=0.2
Loading/Reloading compressibility ratio $\dfrac{m_v(OC)}{m_v(NC)}$	α	[-]	=0.2

5.7.5.3 Results

Figure 5.81 compares between the settlement with time in Rafsanjan plain for 50 [years] obtained by *Ouria* and *Toufigh* (2010) and those from using *LEM* in *GEO Tools*. The results of the *LEM* are almost applicable to the results of *Ouria* and *Toufigh* (2010). In the first 20 cycles, the results of the *LEM* are little greater than the results of *Ouria* and *Toufigh* (2010) with maximum difference not exceed 4 [%]. After that the results of the two models are almost identical.

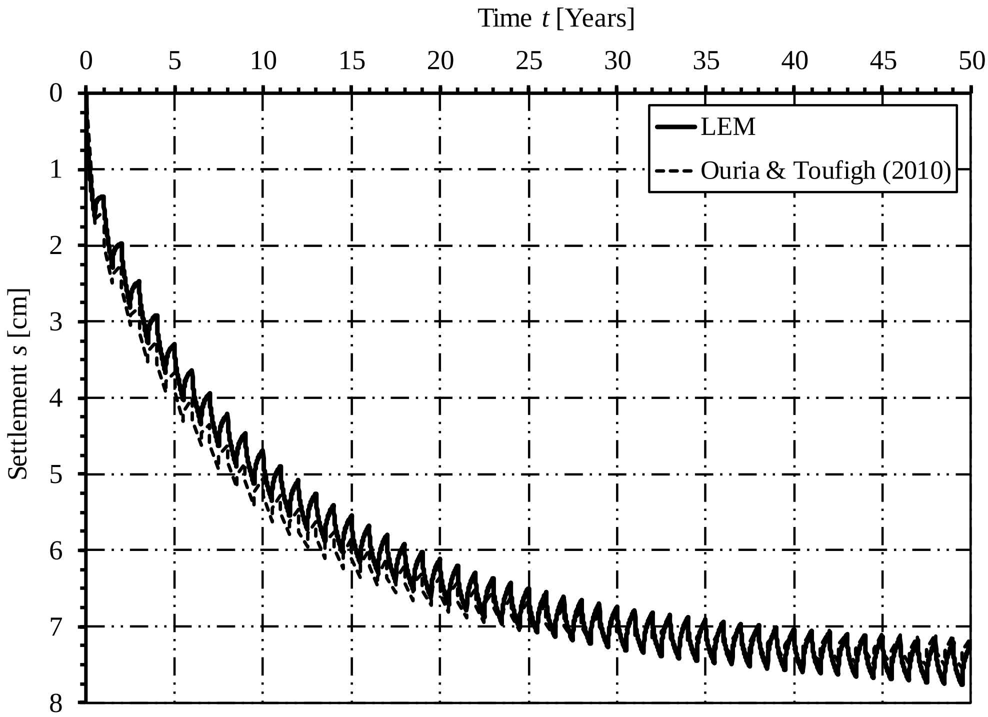


Figure 5.81 Settlement *s* with time *t* in Rafsanjan plain

5.7.5.4 Degree of consolidation by GEO Tools

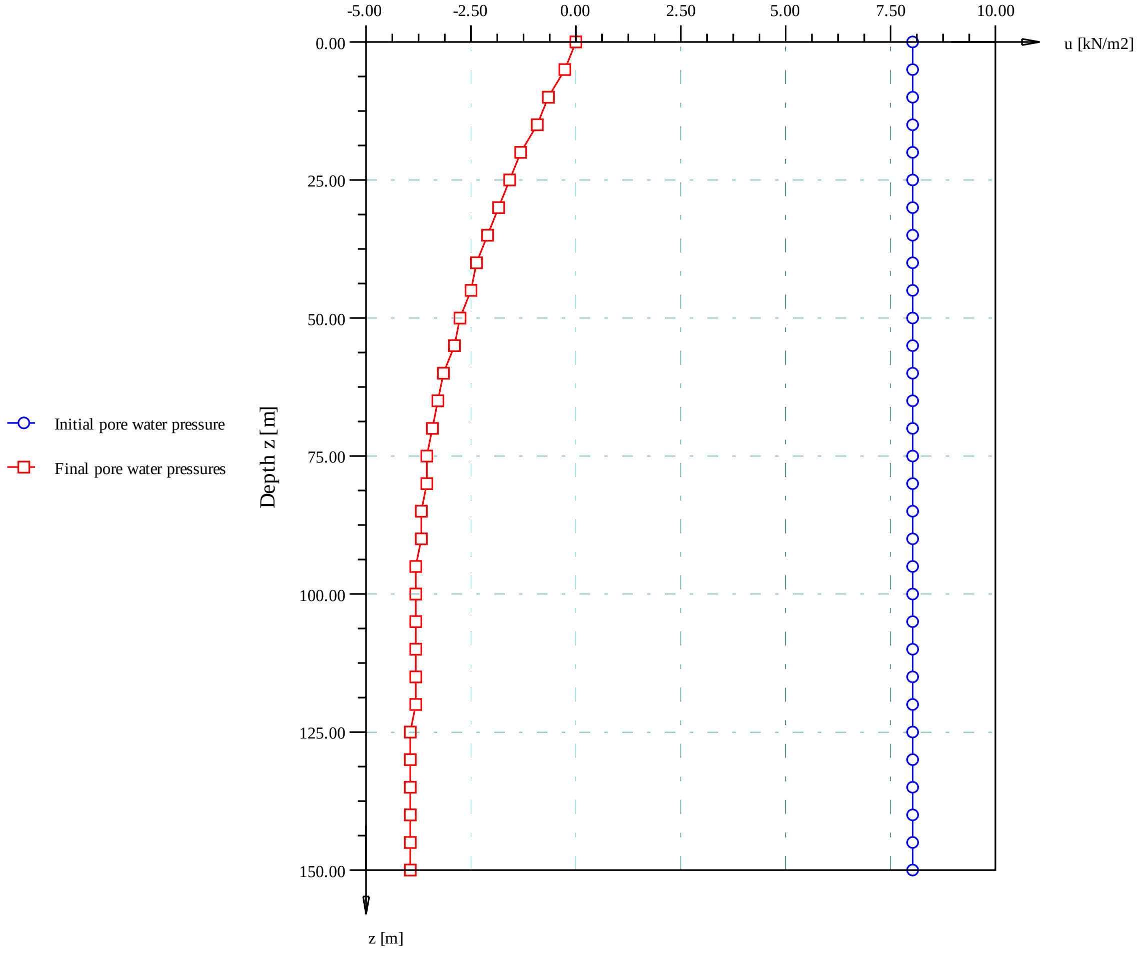
The input data and results of *GEO Tools* for this example are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Prediction of Land Subsidence
Date: 29-09-2017
Project: Ouria and Toufigh (2010)- Rafsanjan
File: Ouria & Toufigh (2010)

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Rectangular cyclic loading
Drainage conditions: Impervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure         uo          [kN/m2] = 8.00
Overburden pressure                  Po=Gamma*z  [kN/m2] = 0.00

Point coordinates/ Layers:
Layer thickness                      Hb          [m]     = 150.00
Depth increment in z-direction       Di          [m]     = 5.00

Time:
Time of consolidation                Tr          [Years] = 50.00
Time increment                       dT          [Years] = 0.01
Time                                 T1          [Years] = 0.00
Time                                 T2          [Years] = 0.50
Time                                 T3          [Years] = 0.00
Time                                 T4          [Years] = 0.50
Period of time                       Tp          [Years] = 1.00
No. of periods                       Np          [Years] = 50

No. of time intervals                Nt          [-]     = 5000
Time interval                        Ti          [Years] = 0.01

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1     150.00         30      2.0602E-05      2.0083E-08
------------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.200
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 0.200

Results:
Degree of consolidation                       Up   [%]  = 36.60
Degree of consolidation                       Us   [%]  = 36.60
Settlement                                    s    [cm] = 7.2300
```

Initial and Final pore water pressures with depth:

No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
0	0.00	8.00	0.00
1	5.00	8.00	-0.32
2	10.00	8.00	-0.64
3	15.00	8.00	-0.96
4	20.00	8.00	-1.26
5	25.00	8.00	-1.55
6	30.00	8.00	-1.83
7	35.00	8.00	-2.09
8	40.00	8.00	-2.34
9	45.00	8.00	-2.56
10	50.00	8.00	-2.76
11	55.00	8.00	-2.95
12	60.00	8.00	-3.11
13	65.00	8.00	-3.26
14	70.00	8.00	-3.38
15	75.00	8.00	-3.49
16	80.00	8.00	-3.58
17	85.00	8.00	-3.65
18	90.00	8.00	-3.72
19	95.00	8.00	-3.76
20	100.00	8.00	-3.80
21	105.00	8.00	-3.83
22	110.00	8.00	-3.85
23	115.00	8.00	-3.86
24	120.00	8.00	-3.87
25	125.00	8.00	-3.88
26	130.00	8.00	-3.88
27	135.00	8.00	-3.89
28	140.00	8.00	-3.89
29	145.00	8.00	-3.89
30	150.00	8.00	-3.89

Pore water pressure u [kN/m2]



Pore water pressure

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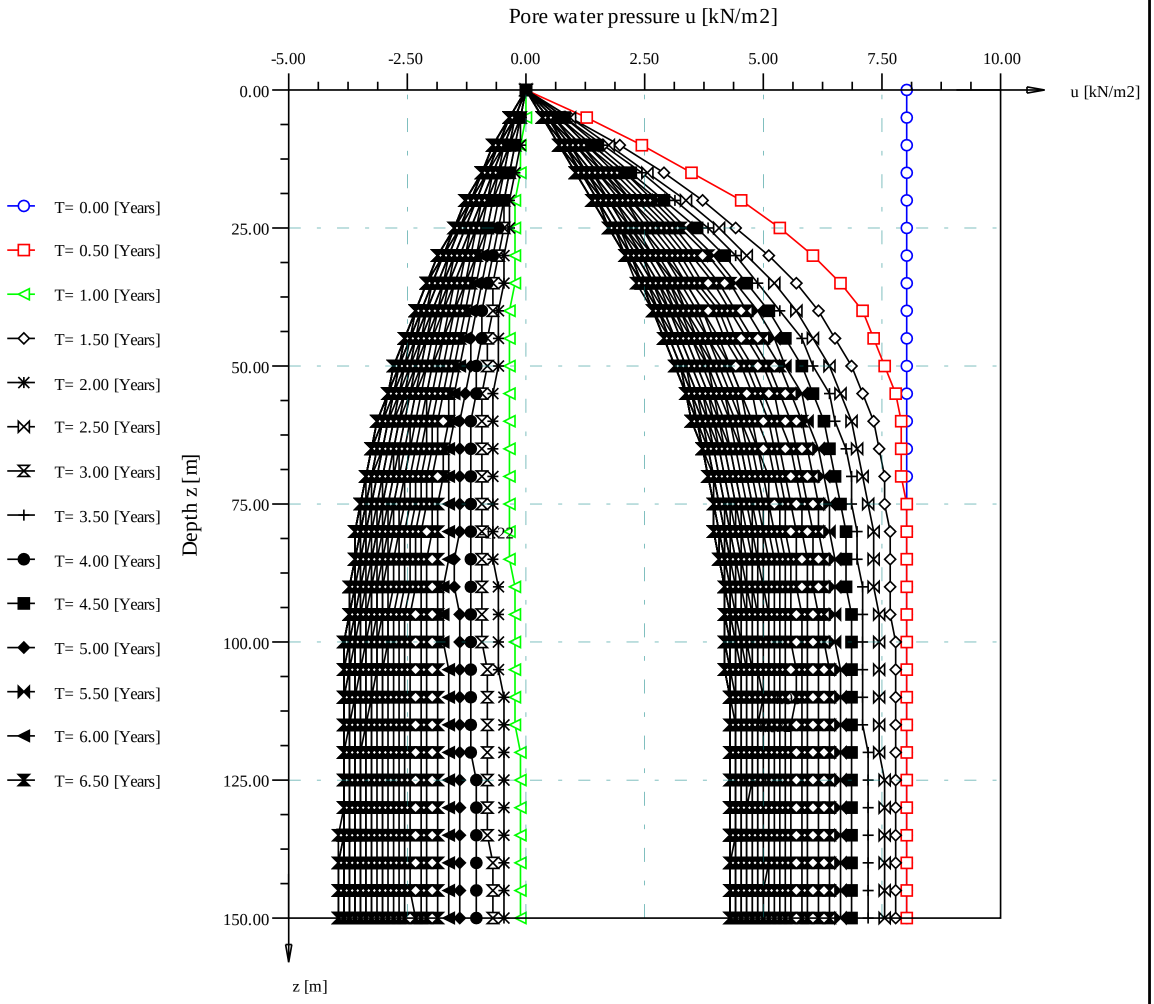
Scale: 131	**Project: Ouria and Toufigh (2010)- Rafsanjan**
File: Ouria & Toufigh (2010)	**Date: 29-09-2017**
Page No.:	**Title: Prediction of Land Subsidence**



Pore water pressure-Time

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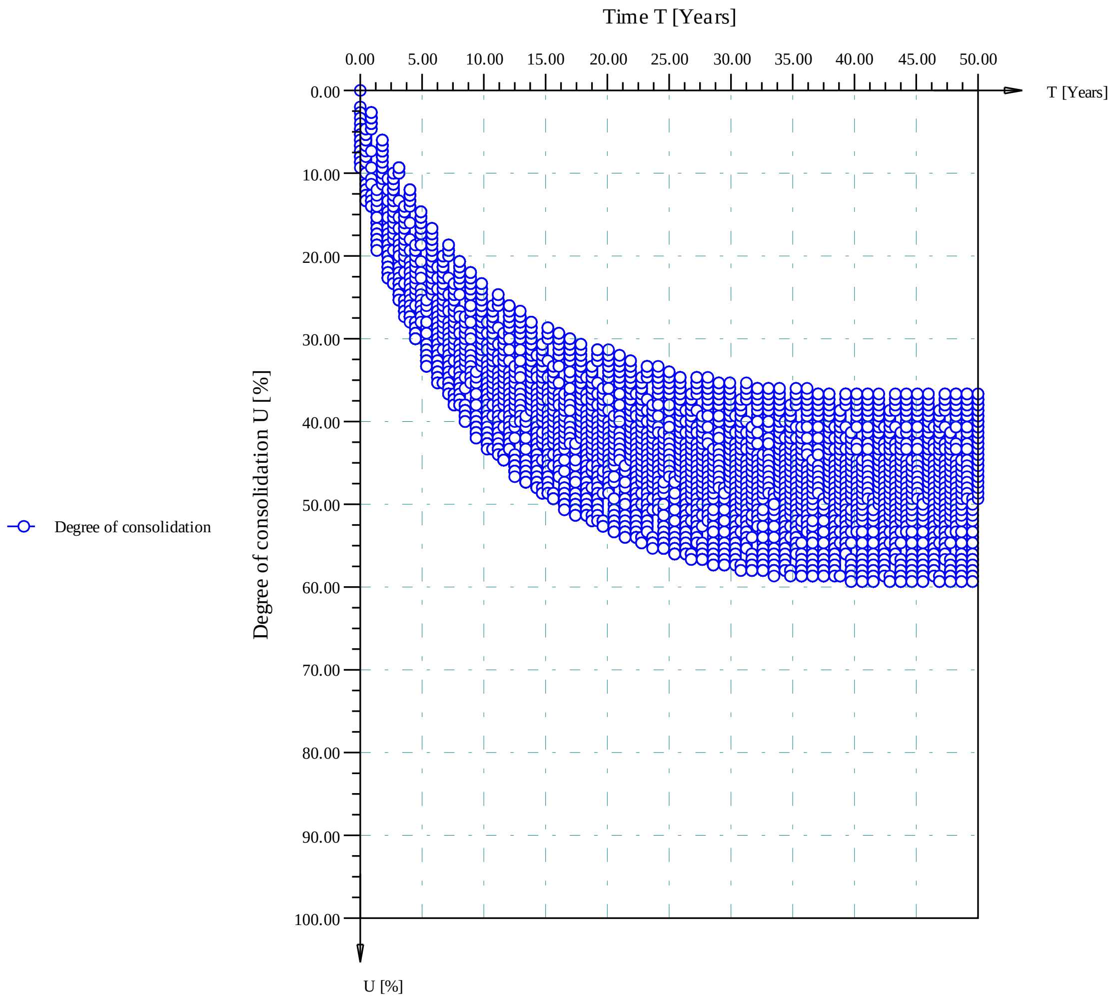
Scale: 116 **File: Ouria & Toufigh (2010)** **Page No.:**	**Project: Ouria and Toufigh (2010)- Rafsanjan** **Date: 29-09-2017** **Title: Prediction of Land Subsidence**



Degree of consolidation-Time

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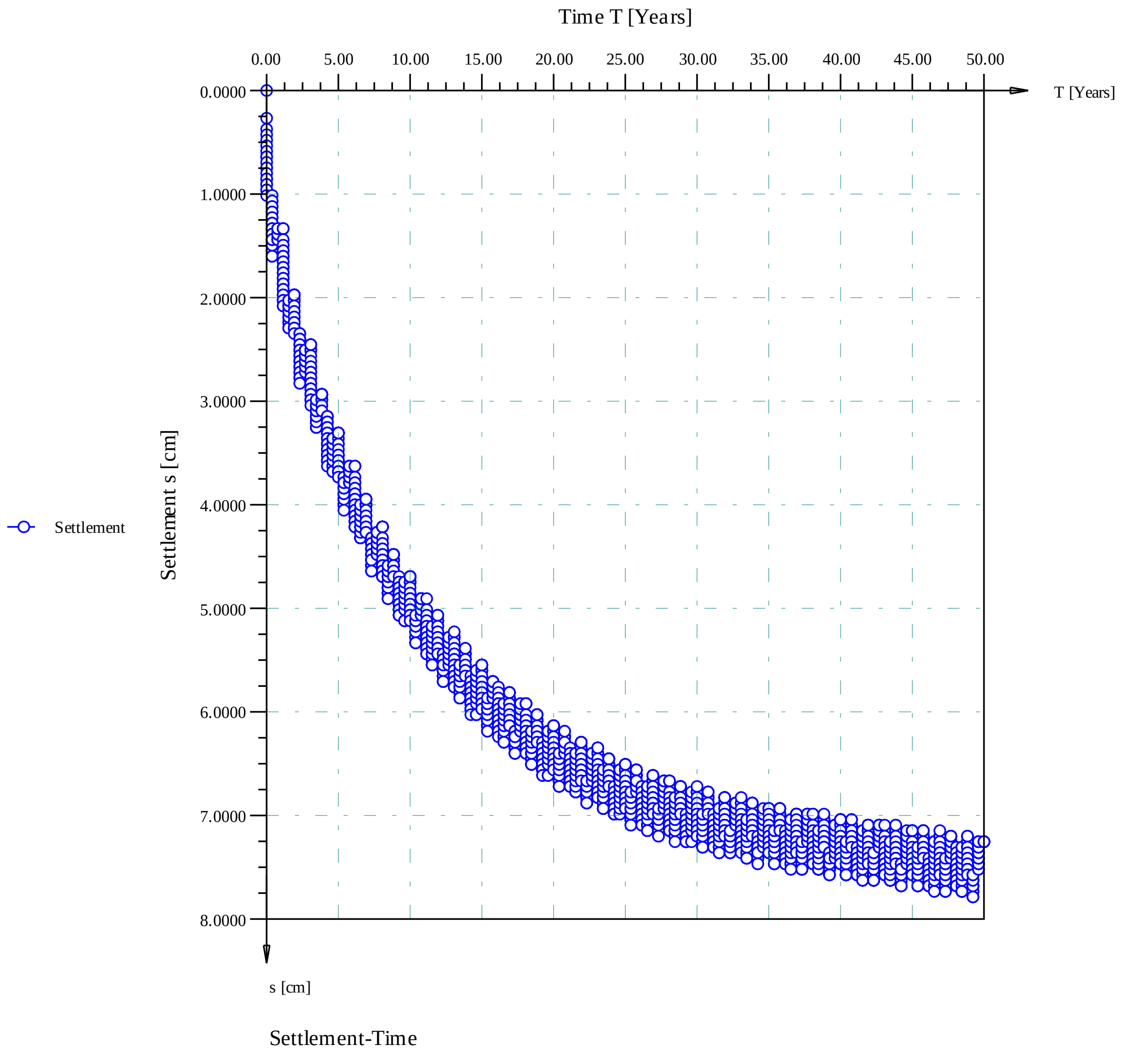
Scale: 444	Project: Ouria and Toufigh (2010)- Rafsanjan
File: Ouria & Toufigh (2010)	Date: 29-09-2017
Page No.:	Title: Prediction of Land Subsidence



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PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 386	**Project: Ouria and Toufigh (2010)- Rafsanjan**
File: Ouria & Toufigh (2010)	**Date: 29-09-2017**
Page No.:	**Title: Prediction of Land Subsidence**

5.7.6 Example 17: Measured Settlement of Clay under Rectangular Cyclic Loading

5.7.6.1 Description of the problem:

Toufigh and *Ouria* (2009) performed a series of laboratory tests to investigate the consolidation of inelastic clays under cyclic loading using their procedure "Virtual time method (*VTM*)". *Ouria et al.* (2015) choose two samples of them to verify their method "Disturbed state concept (*DSC*)". The first sample in the reference of *Toufigh* and *Ouria* (2009) is chosen to be verified using the *LEM*. The selected sample is a double drainage clay layer subjected to a rectangular cyclic loading with the data shown in Table 5.18.

Table 5.18 Data of sample number 1

Rectangular cyclic load	q_c	= 25	[kN/m^2]
Full cycle period	t_c	= 60	[min]
Thickness of the clay layer	H	= 1.3845	[cm]
Coefficient of consolidation in the NC state	$C_{v(\mathrm{NC})}$	= 0.0012	[cm^2/ min]
Coefficient of volume change in the NC state	$m_{v(\mathrm{NC})}$	= 1.2712	[cm^2/ kN]
Change in coefficient of consolidation ratio	β	= 0.095	[-]
Change in coefficient of volume change ratio	α	= 0.090	[-]

5.7.6.2 Analysis of the problem

The clay layer is divided into 5 layers each of 0.2769 [cm] thick, and the time is divided into 16200 intervals, each of 0.1 [Min]. For the analysis by *GEO Tools*, it is convenient to convert the unit system of the time period to days. For the same time factor, the coefficient of consolidation can be obtained from:

$$T_v = \frac{c_v[\mathrm{cm}^2/\mathrm{min}] \times t_p[\mathrm{min}]}{H_d^2[\mathrm{cm}]} = \frac{C_v[\mathrm{m}^2/\mathrm{day}] \times t_p[\mathrm{day}]}{H_d^2[\mathrm{m}]}$$

$$T_v = \frac{0.0012\ [\mathrm{cm}^2/\mathrm{min}] \times 30\ [\mathrm{min}]}{(1.3845)^2[\mathrm{cm}]} = \frac{C_v[\mathrm{cm}^2/\mathrm{day}] \times 30\ [\mathrm{day}]}{(1.3845)^2[\mathrm{cm}]}$$

$$C_v = 0.0012\ [\mathrm{cm}^2/\mathrm{day}] = 1.3889 \times 10^{-12}\ [\mathrm{m}^2/\mathrm{sec}]$$

Consequently, the coefficient of permeability is obtained from:

$$C_v[\mathrm{m}^2/\mathrm{sec}] = \frac{k_v[\mathrm{m}/\mathrm{sec}]}{\gamma_w[\mathrm{kN}/\mathrm{m}^3] \times m_v[\mathrm{m}^2/\mathrm{kN}]}$$

$$1.3889 \times 10^{-12}\ [\mathrm{m}^2/\mathrm{sec}] = \frac{k_v[\mathrm{m}/\mathrm{sec}]}{9.81\ [\mathrm{kN}/\mathrm{m}^3] \times \left(\frac{1.2712}{10000}\right)\ [\mathrm{m}^2/\mathrm{kN}]}$$

$$k_v = 1.732 \times 10^{-15}\ [\mathrm{m}/\mathrm{sec}]$$

Then, the equivalent example data with the new unit system will be:

Initial pore water pressure	u_o	[kN/m^2]	= 25
Total layer thickness	H_d	[m]	= 0.013845
Depth increment in z-direction	D_i	[m]	= 0.002769
Coefficient of consolidation	C_v	[m^2/ sec]	= 1.3889×10^{-12}
Coefficient of permeability	k_v	[m/ sec]	= 1.732×10^{-15}
Period of time	t_p	[day]	= 60
Time increment	dt	[day]	= 0.1
Loading/Reloading consolidation ratio $\frac{C_v(NC)}{C_v(OC)}$	β	[-]	= 0.095
Loading/Reloading compressibility ratio $\frac{m_v(OC)}{m_v(NC)}$	α	[-]	= 0.09
No. of periods	N_p	[-]	= 27
Pervious bottom boundary			

The settlement due to a rectangular cyclic loading at a period of time t_p as shown in Figure 5.82 is determined at different periods and ploated against that of *Toufigh and Ouria* (2009).

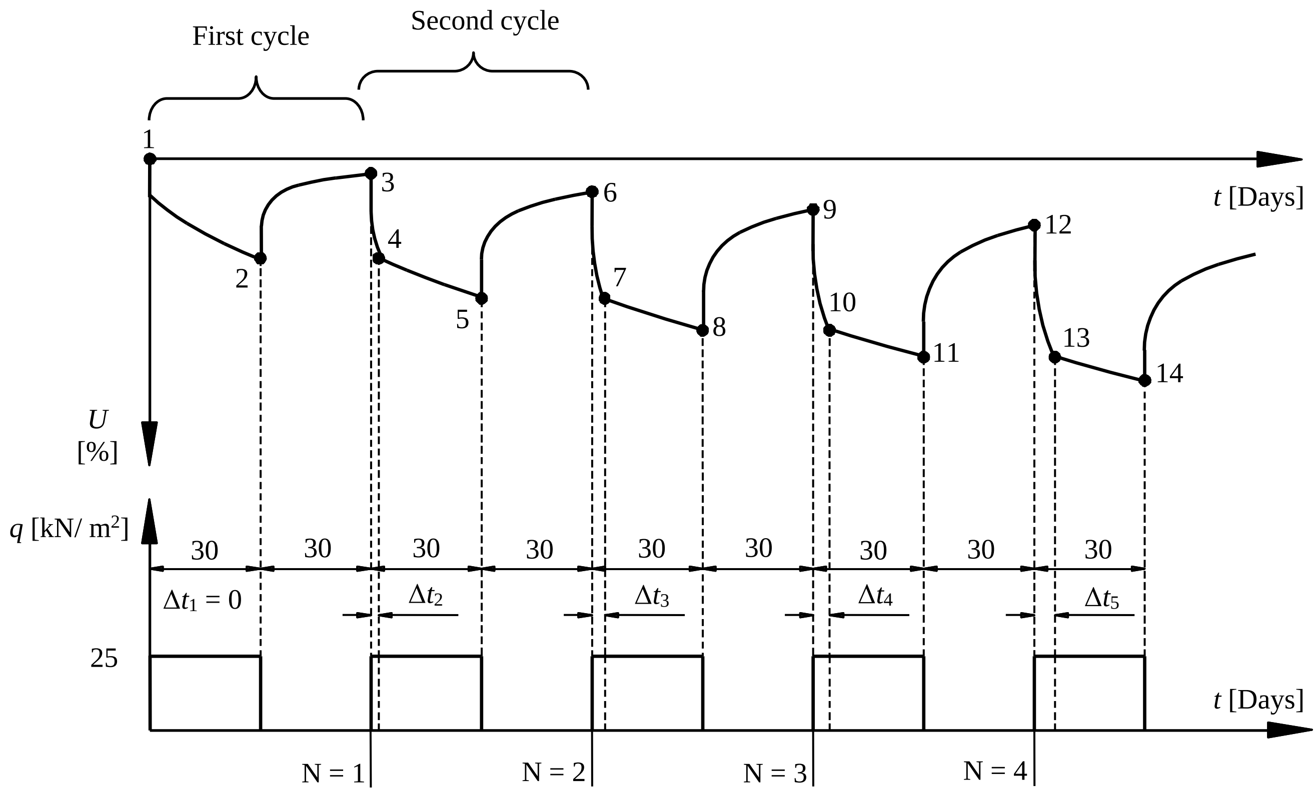


Figure 5.82 Rectangular cyclic loading scheme

5.7.6.3 Results and discussions

As seen in Figure 5.83, which shows the settlement with time for the first sample in the reference of *Toufigh* and *Ouria* (2009), the results of the *LEM* are almost applicable to the results of the *VTM* of *Toufigh* and *Ouria* (2009). The results of the *LEM* are closer to the experimental results than the results of *DSC* of *Ouria et al.* (2015) during the first 10 cycles. Settlements with time of the different analysis methods are applicable to the experimental results. It is noted in this verification example that the *LEM* achieved high efficiency in settlement calculation for this type of clays.

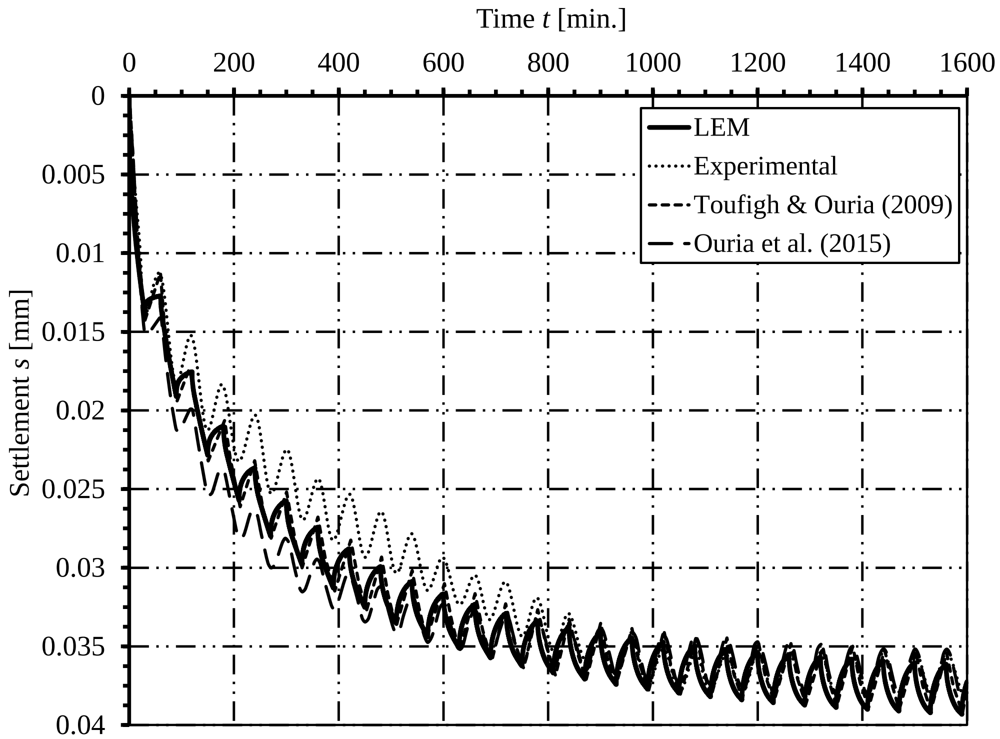


Figure 5.83 Settlement *s* with time *t*

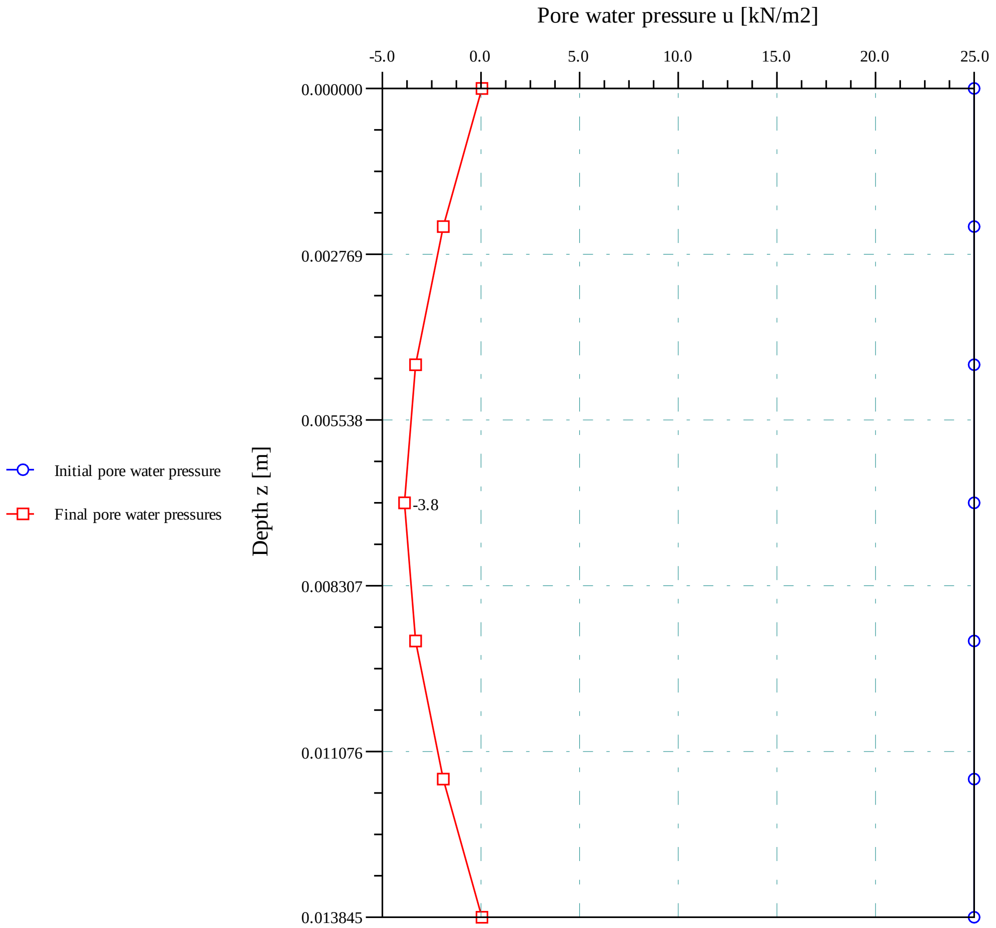
5.7.6.4 Degree of consolidation by GEO Tools

The input data and results of *GEO Tools* for the calculation of the consolidation under a rectangular cyclic loading for this example are presented on the next pages.

```
*************************************************************
                        GEO Tools
                         Version 10
     Program authors Prof. M. El Gendy/ Dr. A. El Gendy
*************************************************************
Title: Consolidation of inelastic clays under rectangular cyclic loading
Date: 19-10-2017
Project:  Toufigh and Ouria 2009
File: Toufigh 2009_1

-------------------------------------------------------------
Degree of consolidation
-------------------------------------------------------------

Method: Layer Equation Method (LEM)
Calculation task: Linear analysis
Loading type: Rectangular cyclic loading
Drainage conditions: Pervious bottom boundary

Initial pore water pressure is:
Constant pore water pressure       uo         [kN/m2] = 25.0
Overburden pressure                Po=Gamma*z [kN/m2] = 0.0

Point coordinates/ Layers:
Layer thickness                    Hb         [m]     = 0.013850
Depth increment in z-direction     Di         [m]     = 0.002769

Time:
Time of consolidation              Tr         [Days]  = 1619.8
Time increment                     dT         [Days]  = 0.1
Time                               T1         [Days]  = 0.0
Time                               T2         [Days]  = 30.0
Time                               T3         [Days]  = 0.0
Time                               T4         [Days]  = 30.0
Period of time                     Tp         [Days]  = 60.0
No. of periods                     Np         [Days]  = 27

No. of time intervals              Nt         [-]     = 16200
Time interval                      Ti         [Days]  = 0.1

Boring:
------------------------------------------------------------
Layer      Layer     No. of  Coefficient of  Coefficient of
  No.  thickness  sublayers   consolidation    permeability
    I          h        Nsl              Cv               k
  [-]        [m]        [-]          [m2/s]           [m/s]
------------------------------------------------------------
    1   0.013850          6      1.3889E-12      1.7320E-15
------------------------------------------------------------
Loading/ reloading ratio Cv(NC)/Cv(OC)        Beta [-]  = 0.095
Loading/ reloading ratio mv(OC)/mv(NC)        Alfa [-]  = 0.090

Results:
Degree of consolidation                       Up   [%]  = 9.59
Degree of consolidation                       Us   [%]  = 9.59
Settlement                                    s    [mm] = 0.0363
```

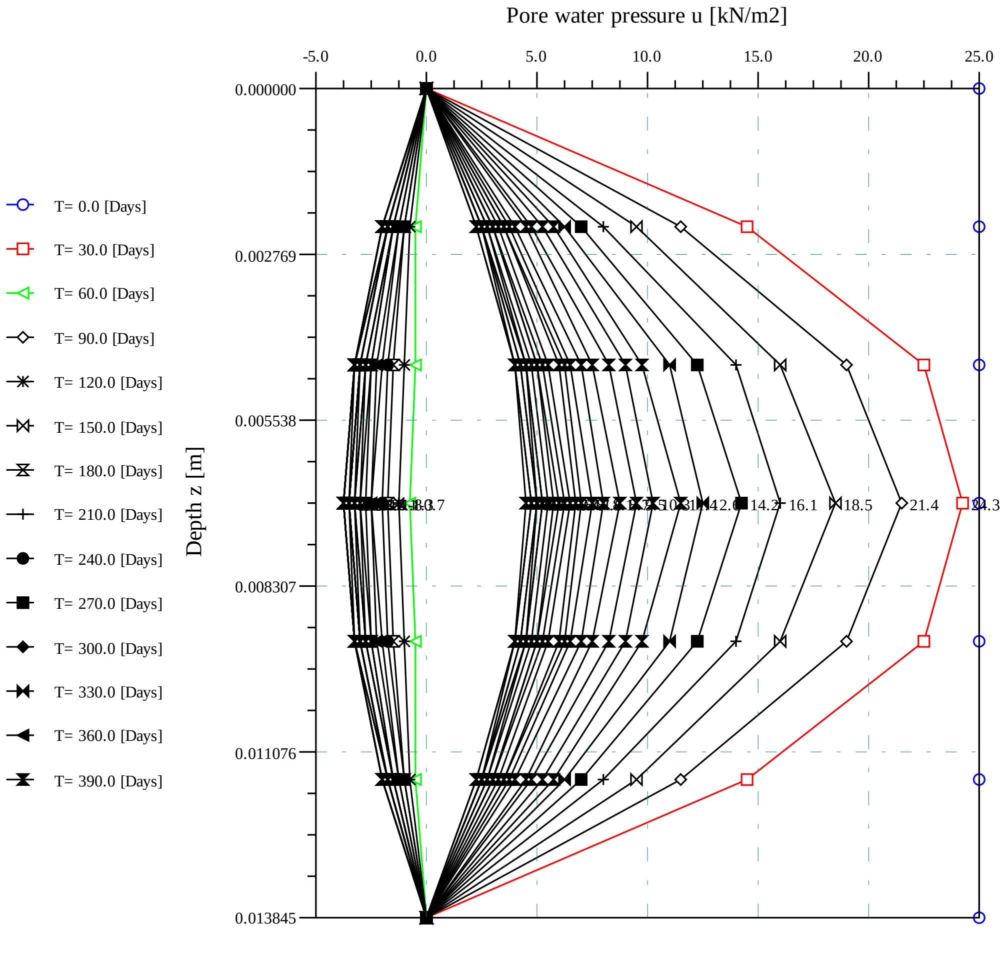
Initial and Final pore water pressures with depth:

No.	Depth	Initial pore water pressure	Final pore water pressures
I	z	uo	uf
[-]	[m]	[kN/m2]	[kN/m2]
0	0.000000	25.0	0.0
1	0.002308	25.0	-1.9
2	0.004617	25.0	-3.3
3	0.006925	25.0	-3.8
4	0.009233	25.0	-3.3
5	0.011542	25.0	-1.9
6	0.013850	25.0	0.0



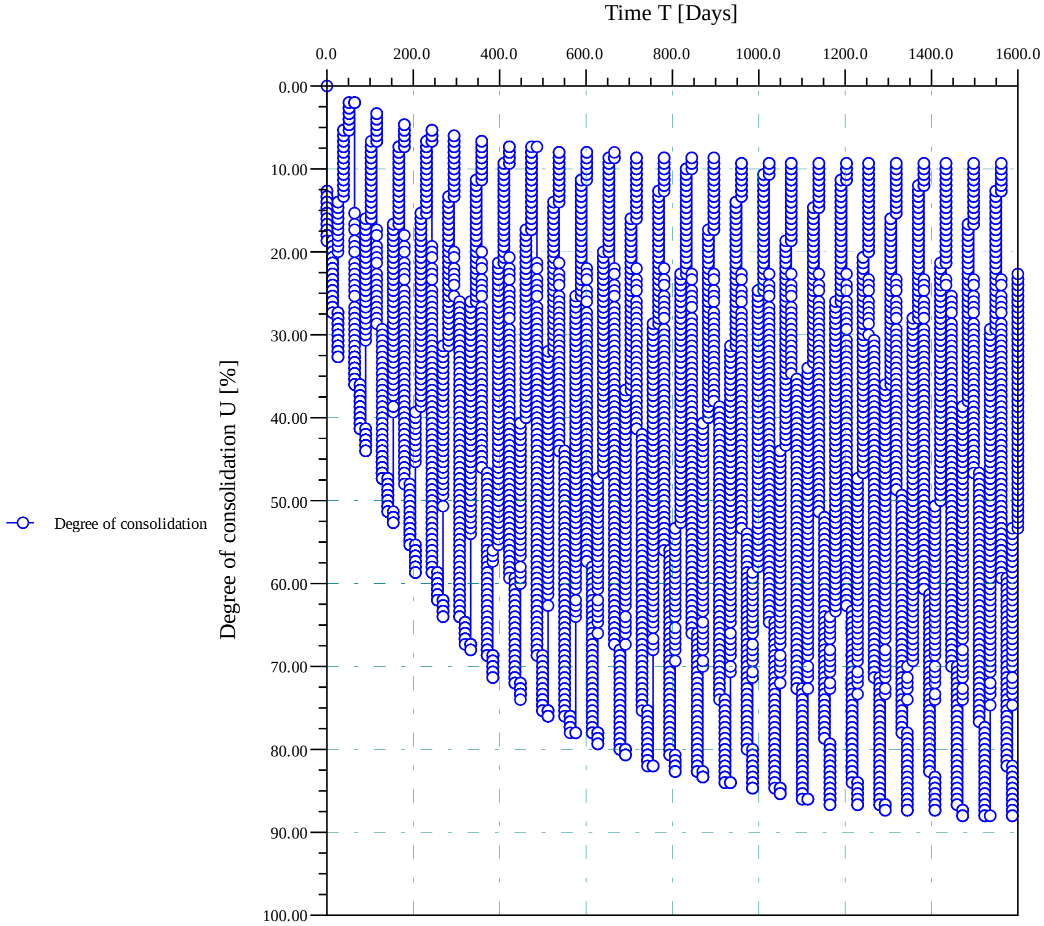
Pore water pressure

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 280	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009_1	**Date: 19-10-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**



Pore water pressure-Time

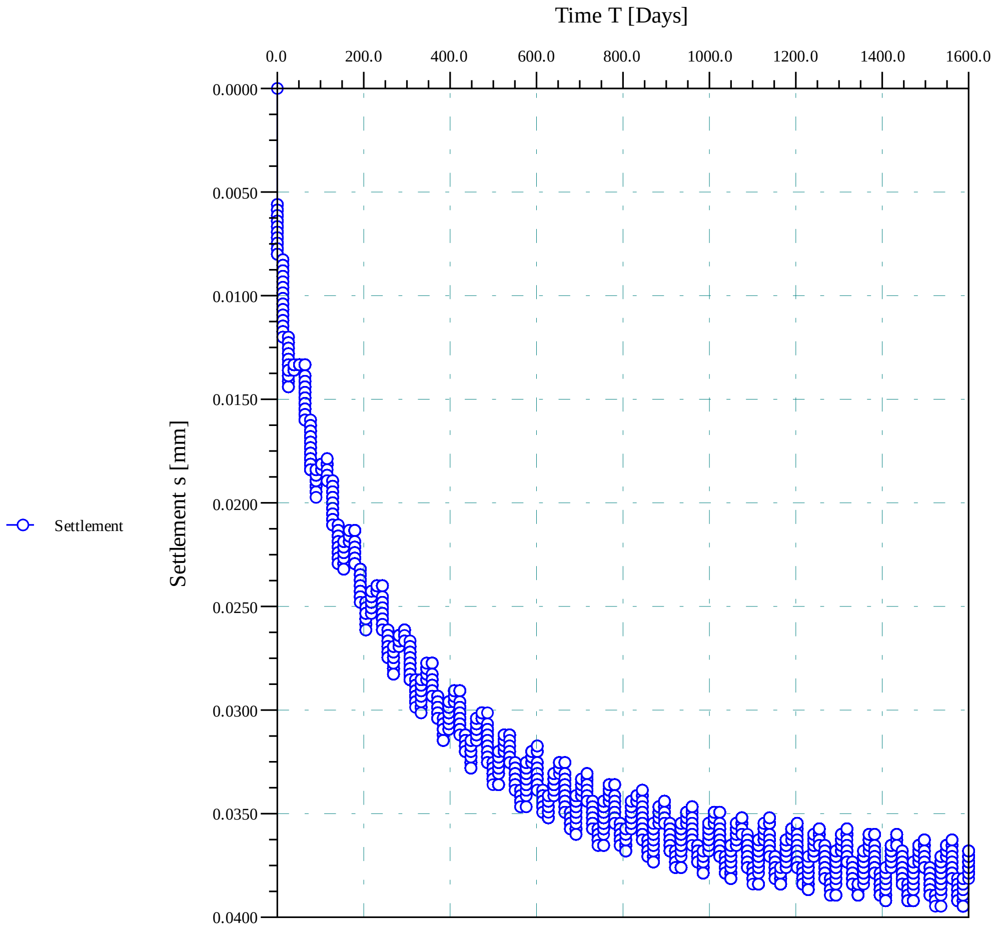
GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 250	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009_1	**Date: 19-10-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**



Degree of consolidation-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7
PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323

Scale: 12800	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009_1	**Date: 19-10-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**



Settlement-Time

GEOTEC Software Inc., Calgary AB, Canada T3E 7Y7 PO Box 14001 Richmond Road PO, Tele.: +1 (587) 332-3323	
Scale: 12800	**Project: Toufigh and Ouria 2009**
File: Toufigh 2009_1	**Date: 19-10-2017**
Page No.:	**Title: Consolidation of inelastic clays under rectangular cyclic loading**

5.8 Notation

The following symbols are used in this paper:

A_j Area around node *j*, [m^2];
A_{ij} Coefficient of basis functions for layer *i*;
B_{ij} Coefficient of basis functions for layer *i*;
C_j Constants of basis functions;
C_C Compression index, [-];
C_{ci} Compressibility index of layer *i*, [-];
C_{ri} Recompression index of layer *i*, [-];
$C_{\alpha i}$ Coefficient of secondary consolidation for layer *i*, [-];
$C_{\alpha ri}$ Reloading coefficient of secondary consolidation for layer *i*, [-];
c_{vi} Coefficient of consolidation of layer *i*, [Year/m^2];
e_i Void ratio at time *t* of layer *i*, [-];
e_{oi} Initial void ratio of layer *i*, [-];
e_{pi} Void ratio for layer *i* at the end of primary consolidation, [-];
$f_{l,j}$ Stress coefficient of node *l* due to contact force at node *j* on the surface, [1/m^2];
H Total thickness of clay layers, [m];
h_i Thickness of layer *i*, [m];
h_{pi} Thickness of the compressible soil layer *i* at time t_p, [m];
k_{vi} Coefficient of permeability of layer *i*, [m/Year];
k_{voi} Initial coefficients of permeability in a layer *i*, [m/year];
M Number of steps-load increment, [-];
m_{vi} Coefficient of volume change of layer *i*, [m^2/kN];
m_{voi} Initial coefficients of volume change in a layer *i*, [m^2/kN];
m Number of grid nodes, [-];
m_i Number of sub-layers in a layer *i*, [-];
m_{vi} Coefficients of volume change in a layer *i*, [m^2/kN];
m_{ri} Coefficients of volume change for reloading in a layer *i*, [m^2/kN];
N Number of function terms (Number of studied nodes);
Q_j Contact force at node *j*, [kN] ;
q_c Contact pressure at the surface due to construction load, [kN/m^2];
q_j Contact pressure at node *j*, [kN/m^2];
r Total number of studied nodes of clay layers, [-];
s_{si} Secondary consolidation settlement of a layer *i*, [m];
s_{pi} Primary consolidation settlement of a layer *i*, [m];
S_{kt} Sum of primary consolidation settlements in all layers at the required time *t*, [m];
S_{ku} Sum of finial consolidation settlements in all layers, when $\Delta u_i = 0$, [m];
t Time for which excess pore water pressure is computed, [Year];
t_c Construction time, [Year];
t_p Time at the end of primary consolidation, [Year];
T_{vi} Time factor for layer *i*, $T_{vi} = c_{vi}t/H^2$, [-];
$u(z, t)$ Excess pore water pressure at any vertical depth *z* and time *t*, [kN/m^2];
U_p Degree of consolidation at the required time *t* in terms of stress;
U_s Degree of consolidation at the required time *t* in terms of settlement;
z_i Vertical coordinate of layer *i*, [m];

γ_w Unit weight of the water, [kN/m^3];
ξ_i Local depth ratio of layer i, $\xi_i = z_i/h_i$, [-];
λ_j Differential equation operator;
μ_i Parameter of the coefficient of consolidation and thickness of layer i;
σ_l Initial vertical stress in a node depth l, [kN/m^2];
σ'_i Effective stresses at time t of layer i, [kN/m^2];
σ'_{oi} Initial effective stresses at the middle of layer i, [kN/m^2];
σ'_{ci} Pre-consolidation pressure of a layer i, [kN/m^2];
$\varphi_j(z)$ A set of basis functions in the variable z only;
$\psi_j(t)$ Coefficients of basis functions in the variable t only;
ω_j Index of the exponential functions in matrix [E_v];
Δq_i Load increment at variable interval of times, [kN/m^2];
Δt_i Interval of times, $\Delta t = t_c/(M\text{-}1)$, [Year];
Δu_i Average excess power water pressure at time t in layer i, [kN/m^2];
Δu_{oi} Initial average stress in a layer i, [kN/m^2];
Δz_i Depth increment in sub-layer i, [m];
$\Delta\sigma'_i$ Increment of vertical stress at time t in a layer i, [kN/m^2];
$\Delta\sigma'_{ri}$ Reloading increment of vertical stress in a layer i, [kN/m^2];
$\Delta\sigma'_{ei}$ Loading increment of vertical stress in a layer i, [kN/m^2];
$\{C\}$ Vector of constants C_j, j=1 to N;
$[D]$ Diagonal square matrix due to variable loading;
$[E_v]$ Diagonal square matrix of exponential functions;
$\{R\}_n$ Vector o obtained from boundary conditions;
$\{u\}$ Vector of the excess pore water pressure u_j, j=1 to N;
$\{u\}_o$ Vector of initial excess pore water pressure;
$\{\Delta q\}_k$ Vector of applied load at interval k;
$[\Phi]$ Matrix of basis functions.

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